Special Topics on Algorithms
Introduction

Vangelis Markakis

Special Topics on Algorithms

- A continuation of the Algorithms course
- Emphasis on topics not covered during the Algorithms course and also on some more modern topics and applications
- You can take this course during your 3rd year or later
- Prerequisites:
  - You have passed the Algorithms course
  - You liked the Algorithms course

Content – Topics to be covered

- Introduction
  - Some basic concepts
  - Distinction between polynomial, pseudopolynomial and exponential time algorithms
- Problems on numbers
  - Exponentiation/Fibonacci/Euclid’s Algorithm for GCD
  - Modular arithmetic, prime numbers, primality testing
  - Applications: public key cryptosystems, RSA and digital signatures

Content – Topics to be covered

- Average case analysis
  - Sorting: Insertionsort, Quicksort
  - Binary Search Trees, hashing
- Coping with NP-completeness – Approximation algorithms
  - Greedy and other combinatorial algorithms
    - Vertex Cover, Set Cover, Maximum Coverage, TSP
    - Partition, Knapsack, Scheduling, Bin Packing
    - SAT
- Randomized Algorithms
  - Max Cut, Min Cut, Max k-SAT

Content – Topics to be covered

- Flows and Matchings
  - Algorithms for the Maximum Flow in a network graph and the Maximum Matching in bipartite graphs.
- (Integer) Linear Programming
  - Applications and LP based Approximation Algorithms
  - LP duality
- Invited lectures
  - We may have 2 lectures by other faculty members and collaborators on some applications

Bibliography

- [DPV] S. Dasgupta, C. H. Papadimitriou, U. V. Vazirani : "Algorithms"
- [CLRS] T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein: "Introduction to Algorithms"
- [KT] J. Kleinberg, E. Tardos: "Algorithm Design"
- …
- **and many resources on the WWW**

Communication

- Office hours:
  - Tuesdays: 12:00 – 14:00
  - Fridays: 13:00 – 14:00

- You can always email me regarding questions
  - If I do not reply within 3 days, send it again

- Eclass: Ειδικά Θέματα Αλγορίθμων
  - Please check the announcements there at least once per week

Tutorials

- Teaching Assistant: Panagiotis Tsamopoulos

- Office hours for the TA to be announced soon

- Tutorials starting next week

Grading

Final exam	**75%**
Midterm exam	**20%**
Individual Assignments (x2)	**15%**

Note: The midterm is used only if it helps your final grade, otherwise the final exam will count as 95%

Date of midterm: (probably) first week of December

Introductory concepts:
Polynomial, Pseudo-Polynomial and Exponential Algorithms

What are we interested in?

Problems to be solved by a machine: precisely defined; no ambiguities

- We want to transform appropriately the input data (problem instances) to output data
- Two subcategories are decision and optimization problems.

COMPUTATIONAL PROBLEM

A problem where we are given **input** instances and some computational question and we want to find an answer/**output**:

E.g., given a graph we wish to compute the set of vertices of odd degree, or to compute a set of k vertices where every pair of them is connected by an edge.

Examples of Problems

EXP(onentiation)

I: positive integers a,n
Q: calculate a^n

FIBONACCI NUMBERS

I: a positive integer n
Q: calculate the n-th Fibonacci number F_n

SUBSET SUM

I: a set $S=\{a_1, a_2, \dots, a_n\}$ of n positive integers and an integer B
Q: is there a subset $A \subseteq S$ s. t. $\sum_{i\in A} a_i = B$?

SAT(isfiability)

I: a boolean formula ϕ
Q: Is ϕ satisfiable ?
(is there a value assignment to its variables making ϕ TRUE ?
= truth assignment)

Algorithms

Three crucial questions about any algorithm for any problem:

1. **Is it correct ?**
   - Does it always terminate?
   - Does it give a correct answer for any instance of the problem ?
2. **How much time/space does it take, as a function of its input?**
   - “time” = number of steps / “space” = number of bits in memory
   - “time” independent of language/implementation/machine
   - We mostly focus on time, expressed as a function T(n), where n is the size of the instance we try to solve
   - Interested in asymptotic behavior of T(n)
   - Notation: O, Ω, Θ, o, ω
3. **Can we do better ?**

Time Complexity of an algorithm

There are many instances of the same size

How does the algorithm work over all these instances?

Best-case complexity

- The minimum number of steps taken on any instance of size n
- Not useful, too optimistic

Worst-case complexity

- The maximum number of steps taken on any instance of size n
- An upper bound on the complexity of the problem
- The most usual analysis

Average case complexity

- The average number of steps taken on any instance of size n
- Depends on the distribution of instances (use of probabilities)

Time Complexity of a problem and lower bounds

Complexity of a problem Π: $T_{\Pi}(n)$

The (worst case) complexity of the best (known) algorithm A

$$T_{\mathrm{P}}(n) = \min_{A}\{T_A(n)\}$$

Obtaining a lower bound on a problem's complexity $L_{\Pi}(n)$:

- By proving that <u>there is no</u> algorithm with $T_A(n) < L_{\Pi}(n)$
- Rare results (e.g., log(n!) for sorting).

Optimal algorithm

- An algorithm A, for which $T_A(n) = L_{\Pi}(n)$
- For many problems we still do not know if we have found an optimal algorithm
- Even for well-studied problems, new improvements arise over the years

Algorithm Analysis

- Evaluation of time complexity
  - Average, worst, best case
- Appropriate solution depending on the application requirements

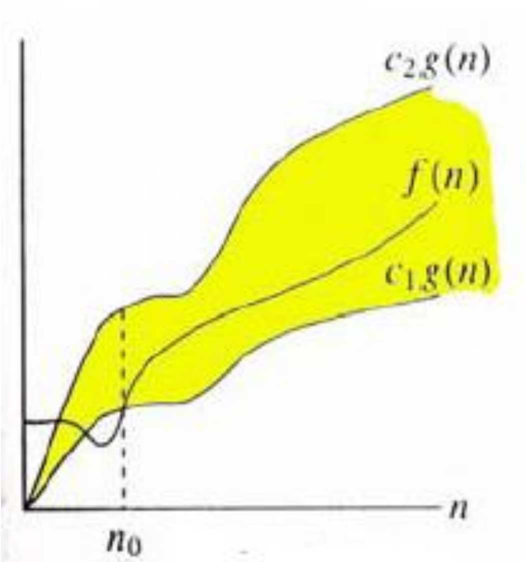
Benefits of theoretical analysis:

- Do not require experimental evaluation but only concrete description of the algorithm
- Results into general conclusions easy to verify, by considering all input instances, determining the time complexity as function of the input size

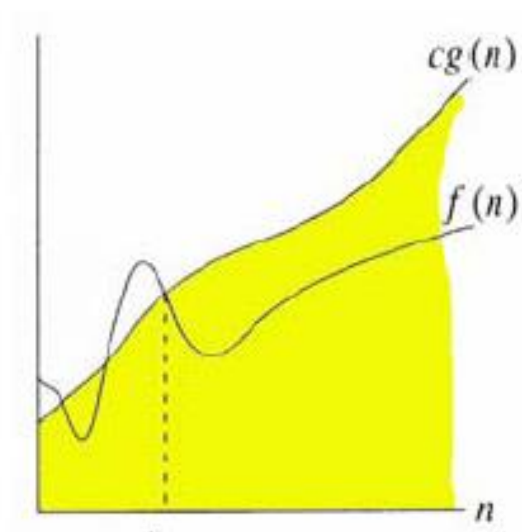
Mathematical background: discrete math (graphs, recurrence relations, combinatorics), mathematical logic, induction in all its forms (simple, strong, structural)

Asymptotic Notation

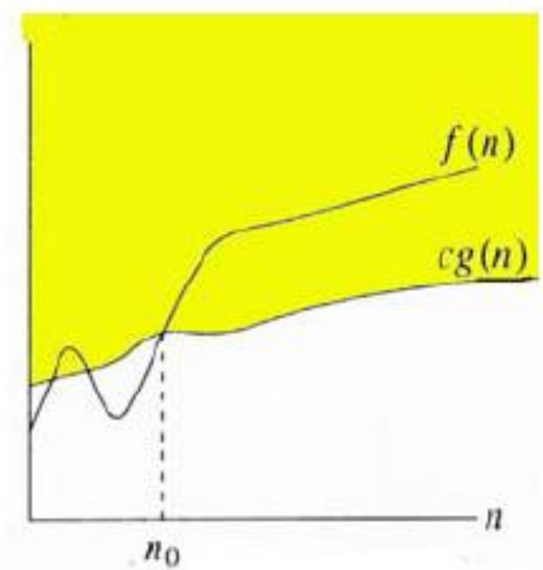
In pictures:



$f(n) = \Theta(g(n))$



$f(n) = O(g(n))$



$f(n) = \Omega(g(n))$

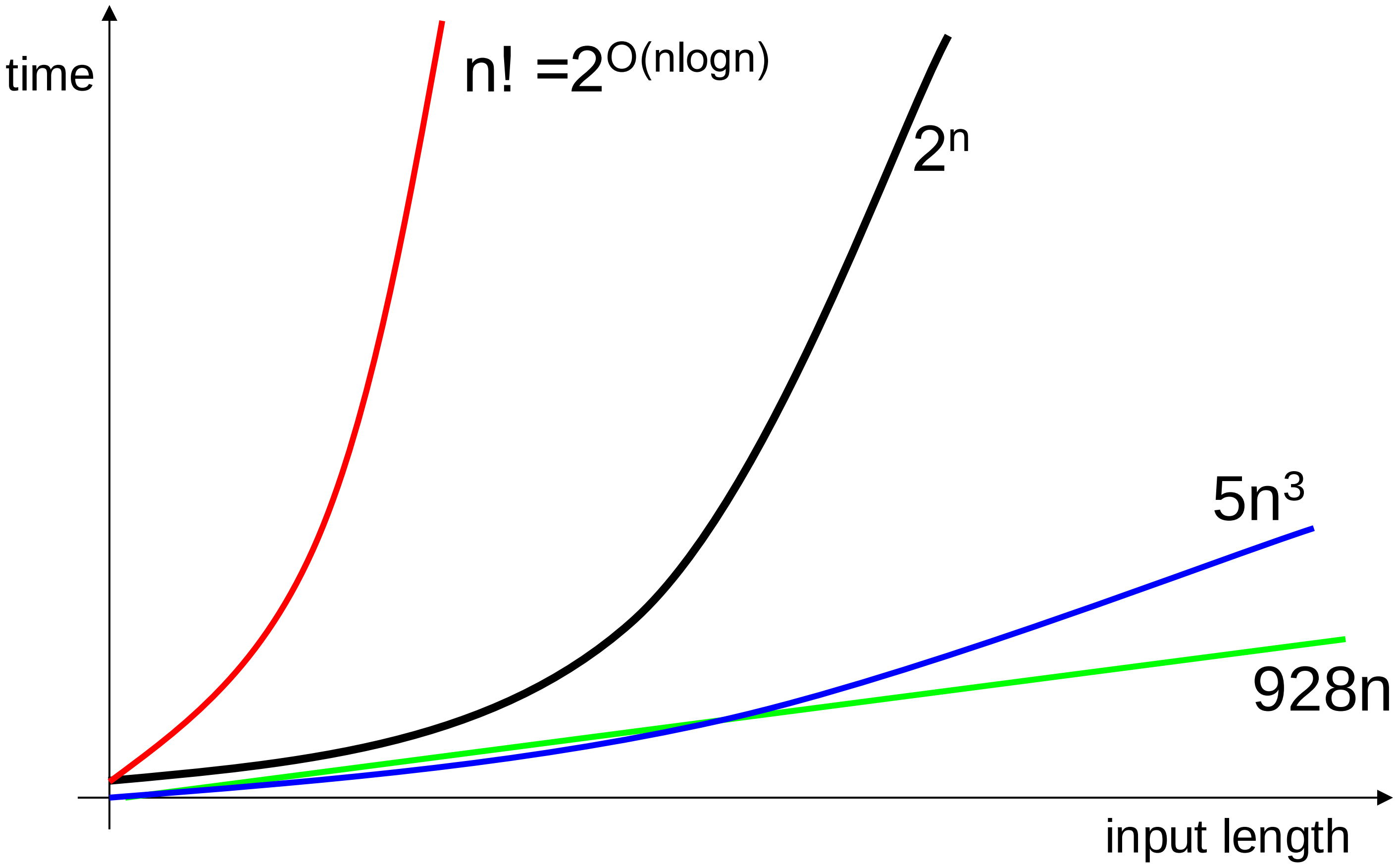
Asymptotic Notation

More formally:

- A function f(n) is O(g(n)) if there exist positive constants c_0 and n_0 such that $f(n) \leq c_0 g(n)$ for every $n \geq n_0$
  - The constant c_0 might be large (but still constant, **independent of** n)
  - Examples:
    - 2n + 10 is O(n). It suffices to set $c_0 = 3$ and $n_0 = 10$
    - 4nlogn + 150n + 3000sqrt(logn) = O(nlogn). Set $c_0 = 3154$, $n_0 = 1$

- A function f(n) is Ω(g(n)) if there exist positive constants c_0 and n_0 such that $f(n) \geq c_0 g(n)$ for every $n \geq n_0$
- A function f(n) is Θ(g(n)) if f(n) is O(g(n)) and f(n) is Ω(g(n))

Growth of various functions



Size of instance and complexity

Consider the description of an instance (i.e., of all the parameters and constraints)

|I| = length of encoded instance/input

Instance $\xrightarrow[\text{e.g. in decimal / binary / unary}]{\text{encoding}}$ encoded instance I

|I| = # of digits of the encoded input

Integer n:	Decimal	Binary	Unary
# bits :	$\lfloor \log_{10} n \rfloor + 1$	$\lfloor \log_2 n \rfloor + 1$	n

Size of instance and complexity

- We typically use the binary encoding
  - but there are reasons to consider other encodings too in complexity theory
- Hence, unless otherwise stated, $|I|$ = # of bits of the encoded input
- Let also N(I) = the largest number in the input
  - Applicable only for problems that have numeric parameters in their input, like Knapsack
- Classification of algorithms
  - Polynomial algorithms: running time $O(poly(|I|)$
  - Exponential algorithms: running time $\Theta(exp(|I|)$
  - Pseudo-Polynomial algorithms: $\Theta(poly(N(I))$, which in worst case is $\Theta(exp(|I|)$
    - We can say that they are $O(poly(|I|))$ if we consider I encoded in unary ! (i.e, polynomial when N(I) not too large)
    - Example: Knapsack admits a dynamic programming algorithm with running time $O(n^2 v_{max})$, where v_{max} is max value in the instance
    - Only relevant for problems with numeric parameters!
    - Not relevant for SAT

Recap from the Algorithms course: Analyzing Recurrence Relations

The Master Theorem

- How do we analyze recurrence relations?
- There are various methods
- The substitution method:
  - Keep substituting until you guess the solution
  - Use induction to prove it formally

Example: $T(n) = T(n-1) + n$, $T(1) = 1$

- $T(n) = T(n-1) + n$
- $= (T(n-2) + n-1) + n$
- $= T(n-2) + n + n-1$
- $= (T(n-3) + n-2) + n + n-1$
- $= \ldots$
- $= n + n-1 + n-2 +\ldots + 2 + 1 = O(n^2)$

Is there a general result that could be applicable to the recurrence relations we will encounter?

The Master Theorem

If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ for some constants $a > 0$, $b > 1$, $d \geq 0$, then

$$T(n) = \begin{cases} \Theta(n^d), & \text{if } \; d > \log_b a \qquad (b^d > a) \\ \Theta(n^d \log_b n), & \text{if } \; d = \log_b a \qquad (b^d = a) \\ \Theta(n^{\log_b a}), & \text{if } \; d < \log_b a \qquad (b^d < a) \end{cases}$$

- Usually convenient to think of n as a power of b, so that n/b is an integer.
- In many cases of interest, b = 2
- More general versions of this theorem are available as well

The Master Theorem - Examples

- Naive integer multiplication (by divide and conquer)
  - $T(n) = 4T(n/2) + O(n)$
  - $a = 4$, $b = 2$, $\log_b a = \log_2 4 = 2$
  - $d = 1 < 2 = \log_b a$
  - Case (iii) applies: $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(\mathrm{n}^2)$
- Karatsuba's algorithm for integer multiplication
  - $T(n) = 3T(n/2) + O(n)$
  - $a = 3$, $b = 2$, $\log_b a = \log_2 3 = 1.59$
  - $d = 1 < \log_b a$
  - Case (iii) applies again: $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(\mathrm{n}^{1.59})$

The Master Theorem - Examples

- $T(n) = 5T(n/25) + O(n^2)$
  - $a = 5$, $b = 25$, $\log_b a = \log_{25} 5 = 0.5$
  - $d = 2 > 0.5 = \log_b a$
  - case (i) applies: $T(n) = \Theta\left(n^d\right) = \Theta(\mathrm{n}^2)$
- $T(n) = T(2n/3) + O(1)$
  - $a = 1$, $b = 3/2$, $\log_b a = \log_{3/2} 1 = 0$
  - $d = 0 = \log_b a$
  - case (ii) applies: $T(n) = \Theta\left(n^0 \log_{3/2} n\right) = \Theta(\mathrm{logn})$
- $T(n) = 9T(n/3) + O(n)$
  - $a = 9$, $b = 3$, $\log_b a = \log_3 9 = 2$
  - $d = 1 < 2 = \log_b a$
  - case (iii) applies: $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(\mathrm{n}^2)$