ΟΙΚΟΝΟΜΙΚΟ
ΠΑΝΕΠΙΣΤΗΜΙΟ
ΑΘΗΝΩΝ

ATHENS UNIVERSITY
OF ECONOMICS
AND BUSINESS

M.Sc. Program in Computer Science
Department of Informatics

Problems with Sets and Partitions

Vangelis Markakis – George Zois

markakis@gmail.com

georzois@gmail.com

Weighted set problems

SUBSET SUM

I: A set of objects S = {1,…,n}, each with a positive integer weight w_i, i = 1,…,n, and a positive integer W

Q: is there $A \subseteq S$ s.t. $\sum_{i \in A} w_i = W$?

PARTITION

I: A set of objects S = {1,…,n}, each with a positive integer weight w_i, i = 1,…,n

Q: is there $A \subseteq S$ s.t. $\sum_{i \in A} w_i = \sum_{i \in S-A} w_i \left(= \frac{1}{2} \sum_{i \in S} w_i\right)$?

0-1 KNAPSACK

I: A set of objects S = {1,…,n}, each with a positive integer weight w_i, and a value v_i, i=1,…,n, and a positive integer W

Q: find $A \subseteq S$ s.t. $\sum_{i \in A} w_i \leq W$ and $\sum_{i \in A} v_i$ is maximized

Weighted set problems

BIN PACKING

I: A set of objects S = {1,…,n}, each with a positive integer weight w_i, i = 1,…,n, and a positive integer W

Q: find a partition of S into $A_1, \dots, A_m$ s.t. $\sum_{i \in A_j} w_i \le W, j = 1,2,\dots,m$ and m is minimized

i.e., minimize the number of bins to fit the objects

MAKESPAN (P||C_{max})

I: A set of objects S = {1,…,n}, each with a positive integer weight w_i, i = 1,…,n, and a positive integer m

Q: find a partition of S into $A_1, \dots, A_m$ s.t. $\max_{1 \le j \le M} \{\sum_{i \in A_j} w_i\}$ is minimized

Weighted set problems

- All these problems are NP-complete

E.g.:

- SUBSET-SUM $\leq_p$ PARTITION
- PARTITION $\leq_p$ BIN PACKING
- BIN PACKING $\leq_p$ MAKESPAN
- PARTITION $\leq_p$ MAKESPAN

A. SUBSET SUM and PARTITION

SUBSET SUM

SUBSET SUM

I: a set S = $\{a_1, a_2, \ldots, a_n\}$ of n positive integers and an integer B

Q: is there a subset A $\subseteq$ S such that $\sum_{i \in A} a_i = B$?

BRUTE FORCE

- there are 2^n possible combinations of n items (= possible number of subsets one can construct from S)
- Go through all combinations and stop in the first one such that

$$\sum_{i \in A} a_i = B$$

- Report NO otherwise
- Running time: $O(n2^n)$

- Can we do better?

SUBSET SUM

- Let $S_i = \{w_1, w_2, \ldots, w_i)$ [the values of the first i elements]
- IDEA (Dynamic Programming): Compute the sums of all subsets of S_i using the sums of all subsets of S_{i-1} (exclude sums > W)

- Let L be a list of integers
- Notation: L+b = a new list with all elements of L increased by b

  e.g., if L = [1, 2, 3, 5], L+2 = [3, 4, 5, 7]

- Auxiliary method MERGE (L,L')
  - Input: 2 sorted lists of integers, L and L'
  - Output: a sorted list that is the merge of L and L' with no duplicates
  - Complexity $O(|L|+|L'|)$

SUBSET SUM

L_i : list of the sums of all subsets of S_i (keep only sums ≤ W)

```
Algorithm SubsetSum (S,W);
$L_0$=[0];
for i=1 to n do
   $L_i$=MERGE($L_{i-1}$, $L_{i-1}+w_i$);
   Remove from $L_i$ every element > W;
Check if the largest element in L equals W;
```

Example

S={1,4,5}, n=3, W=8

S_0: L_0=[0] L_0+w_1 =[1]

S_1: L_1=[0,1] L_1+w_2 =[4,5]

S_2: L_2=[0,1,4,5] L_2+w_3 =[5,6,9,10]

S_3=S: L_3=[0,1,4,5,6] Answer: NO

Complexity ?

SUBSET SUM

Complexity: O(nW)

At every step, the list we keep has at most W elements

- Not polynomial
- But pseudo-polynomial!

PARTITION

- Tightly related to SUBSET SUM
- SUBSET-SUM $\leq_p$ PARTITION, hence NP-complete as well
- We could use the algorithm for SUBSET SUM, setting $W = ½ \Sigma w_i$
- PARTITION is also a special case of scheduling problems
- To solve PARTITION, think of 2 identical processors, with processing times equal to w_i
- Minimizing the makespan would tell us if there exists a solution to the PARTITION problem

B. Scheduling problems

Scheduling Problems

Any problem where

- We have a set of jobs/tasks
- We have a set of processors
- We want to assign the jobs to the processors so as to optimize some criterion

A plethora of such problems have been studied since the 60s

Variations:

- Criterion to optimize: makespan, throughput, sum of (weighted) completion times,...
- Processors: may be completely unrelated (each with a different speed), identical, or uniformly related (speeds are multiples of each other)
- Jobs: they may have arrival times or deadlines, precedence constraints (cannot execute job i before job j finishes), option for preemption (execute only one part of the job now and continue later with the rest)

Scheduling Problems

We will focus on makespan

MAKESPAN ($P||C_{max}$)

I: A set of objects S = {1,…,n}, each with a positive integer weight w_i, i = 1,…,n, and a positive integer m

Q: find a partition of S into $A_1, \dots, A_m$ s.t. $\max_{1 \le j \le M} \{\sum_{i \in A_j} w_i\}$ is minimized

In other words (in shceduling terminology):

Given a set of n independent jobs $J_1, J_2, \dots, J_n$

And a set of m identical machines $M_1, M_2, \dots, M_m$

Let p_i = the processing time of job i on any machine (i.e., $p_i = w_i$)

Problem: schedule the jobs on the machines

in order to minimize $C_{max} = \max_j \{C_j\}$

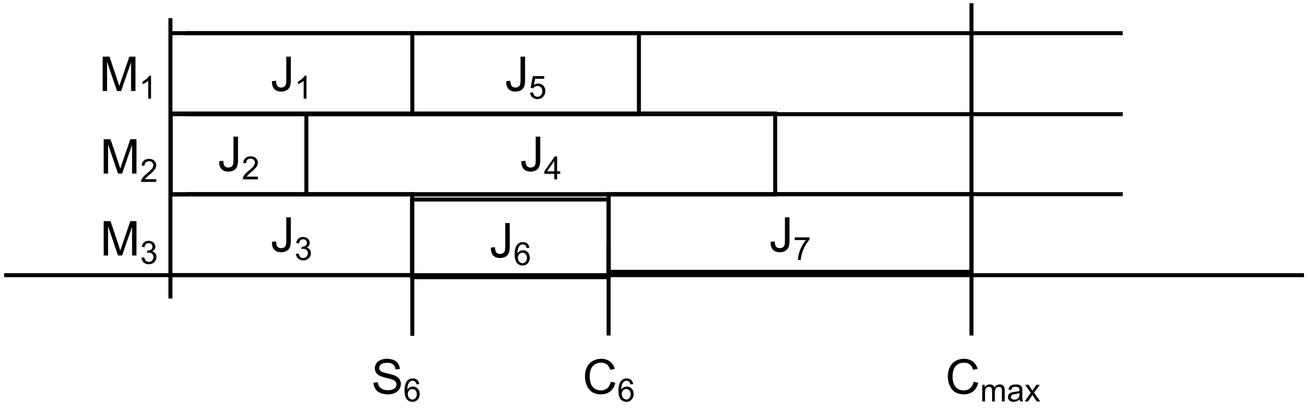
where, for a given assignment of jobs to machines :

C_j : the time job j finishes its execution

Makespan (P | | C_{max})

List scheduling - A general scheduling methodology

- Construct a list of jobs (an ordering of the jobs according to some criterion)
- Whenever a machine becomes available, the next job in the list is scheduled on that machine



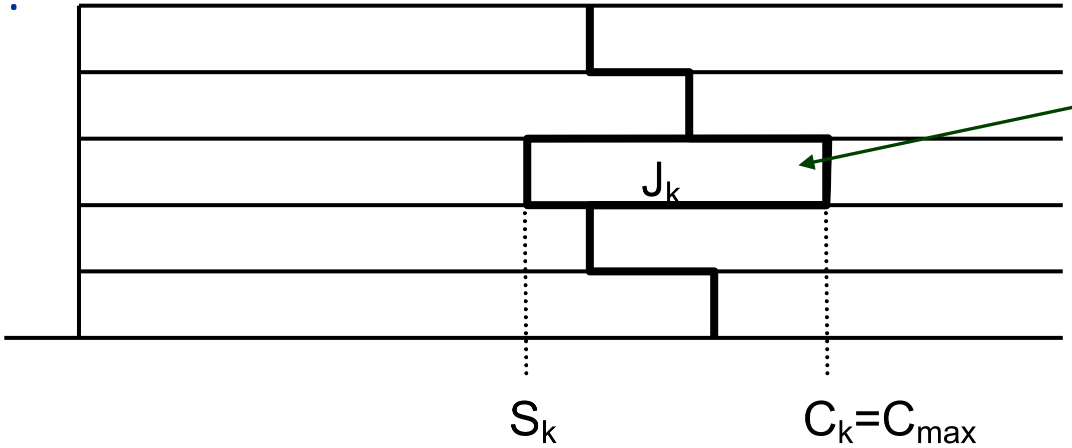
S_j: the time job j starts its execution
C_j: the time job j finishes its execution

Makespan ($P \mid\mid C_{max}$): A 2-approximation

Theorem [Graham, 1966]:

List scheduling using an arbitrary order of the jobs yields a $(2 - 1/m)$-approximation algorithm for $P \mid\mid C_{max}$

Proof:



The job which completes last

$$C_k = S_k + p_k$$

Note that: $$\sum_{i=1}^{n} p_i \geq m\, S_k + p_k \Rightarrow S_k \leq \frac{1}{m}\left(\sum_{i=1}^{n} p_i - p_k\right)$$

and thus, $$C_k \leq \frac{1}{m}\left(\sum_{i=1}^{n} p_i - p_k\right) + p_k = \frac{1}{m}\sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right)p_k \quad (1)$$

Makespan ($P \mid \mid C_{max}$): A 2-approximation

Proof (cont.):

- We need a lower bound for OPT
- We will actually use 2 lower bounds
- Let C* be the makespan of an optimal solution

$$C^* \geq p_k \qquad (2)$$

(in fact the makespan is at least as big as any p_i)

$$C^* \geq \frac{1}{m}\sum_{i=1}^{n} p_i \qquad (3)$$

(best case is if all machines finish at the same time)

Makespan (P | | C_{max}): A 2-approximation

Proof (cont.):

Let C = makespan of the algorithm's solution

$$C = C_k \overset{(1)}{\leq} \frac{1}{m}\sum_{i=1}^{n} p_i + (1 - \frac{1}{m}) p_k$$

$$\overset{(2),(3)}{\leq} C^* + (1 - \frac{1}{m}) C^*$$

Hence: C ≤ (2 - 1/m) C* = (2 – 1/m) OPT

Tightness of ρ=2

Graham's ratio of 2-1/m is tight

Family of instances (parameterized by m, the number of processors):

- m^2 - m + 1 jobs in total
- m^2 – m jobs of processing time 1
- 1 job of processing time m
- m machines

If the long job is last in the list, then $C = ((m^2-m)/m)+m = m-1+m = 2m-1$

The optimal schedule has length $C^* = m$ (why ?)

$C/C^* = (2m-1) / m = 2 - 1/m$

Makespan ($P \mid\mid C_{max}$): A 3/2-approximation

List scheduling using LPT (Longest Processing Time first)

Drawback with previous approach:

- List created arbitrarily
- Long jobs at the end of the list may cause a large makespan

Theorem: List scheduling with the LPT rule yields a 3/2-approximation for $P \mid\mid C_{max}$

Proof:

- Let $p_1 \geq p_2 \geq \dots \dots \geq p_m \geq p_{m+1} \geq \dots \geq p_k \geq \dots \geq p_n$ be the processing times
  - if $n \leq m$, the problem is trivial, so we can safely assume that $n > m$
- Let J_k be the job that finishes last under LPT (say on machine M_i)
- If J_k is the only job on M_i, then the schedule is optimal ($OPT \geq p_k$)
- Otherwise, M_i executes at least two jobs, that is $k \geq m+1$ (the first m jobs go to different machines)

Makespan (P | | C_{max}): A 3/2-approximation

As there are at least m+1 jobs, then two of the first m+1 jobs are executed on the same machine in any optimal schedule (pigeonhole principle)

Hence:

$$C^* \geq 2p_{m+1} \geq 2p_k \Rightarrow p_k \leq \frac{C^*}{2}$$

Using this new bound:

$$C \leq \frac{1}{m}\sum_{i=1}^{n} p_i + (1-\frac{1}{m})p_k \qquad \text{(as before)}$$

$$\leq C^* + (1-\frac{1}{m})\frac{C^*}{2}$$

$$\leq C^*(1+\frac{1}{2}-\frac{1}{2m})$$

$$= C^*(\frac{3}{2}-\frac{1}{2m})$$

Makespan (P | | C_{max}): Can we do better?

Is the (3/2 – 1/2m)-ratio for LPT scheduling tight ?

NO !

The lower bounds we have used in the analysis are too generous!

Theorem [Graham, 1969]:

List scheduling using LPT (Longest Processing Time first) yields a (4/3 – 1/(3m))-approximation algorithm for makespan

Proof a little more involved (omitted here)

Makespan (P | | C_{max}): Tightness for $\rho=4/3$

The (4/3 – 1/3m) ratio for LPT scheduling is tight

Family of instances (parameterized by m, the number of processors):

- 2m + 1 jobs in total
- 2 jobs for each of the weights m+1, m+2, …, 2m-2, 2m-1
  - 2m-2 jobs
- 3 jobs of weight m
- m machines

Example: m = 5

2 jobs of each of the weights 6, 7, 8, 9 (total of 8 jobs)

3 jobs of weight 5 (+ 3 jobs)

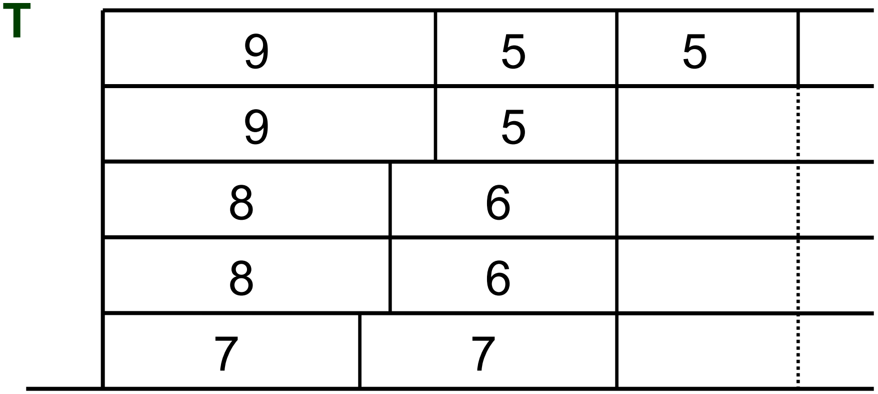
5 machines total of 11 jobs

(= 2m+1)

Makespan ($P \mid\mid C_{max}$): Tightness for $\rho=4/3$

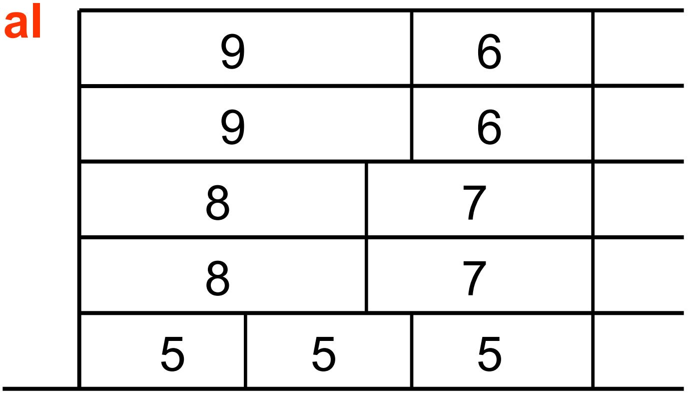
Example (cont.):

LPT



C = 19

Optimal



C* = 15

$$\frac{C}{C^*}=\frac{19}{15}$$

$$\frac{4}{3}-\frac{1}{3m}=\frac{4}{3}-\frac{1}{15}=\frac{19}{15}$$

Makespan (P | | C_{max})

Let us recap on List scheduling

arbitrary list: ρ = 2 – 1/m
LPT: ρ = 4/3 – 1/3m
Can we go beyond 4/3 ?

Another idea for an algorithm

- Find an optimal schedule for the k largest jobs (for some parameter k, to be determined later)
- Schedule the rest of the jobs using the LPT rule

We will find the approximation of this algorithm as a function of k, ρ = f(k)

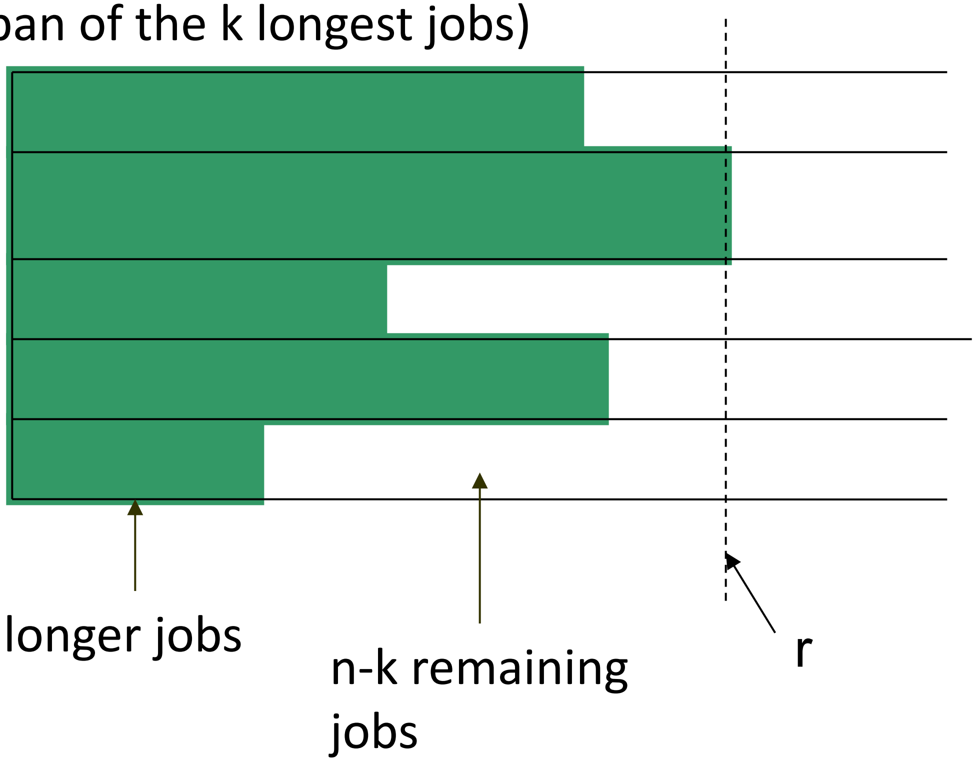
Makespan (P | | C_{max}): ρ=f(k)

Let r be the completion time of the k longest jobs

C = makespan of the algorithm

C* = optimal makespan

Case a) C = r => C = C* (this is the easy case, since C^* ≥ optimal makespan of the k longest jobs)



Makespan ($P \mid\mid C_{max}$): ρ=f(k)

Case b) C > r

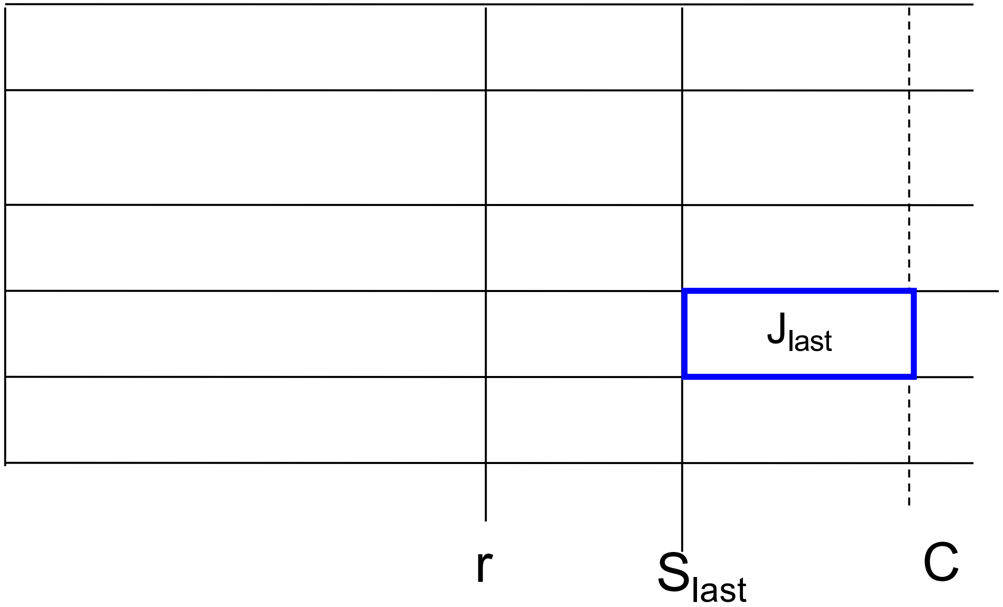
Let J_{last} be the job which finishes last

$p_1 \geq p_2 \geq \ldots \geq p_k \geq p_{k+1} \geq p_{k+2} \geq \ldots \geq p_{last} \geq p_{last+1} \geq \ldots \geq p_n$, $n > m$

(if n<m, the problem is trivial)

$last \geq k+1 \Rightarrow p_{last} \leq p_{k+1}$

All machines are busy till time S_{last} => $m\, S_{last} + p_{last} \leq \sum_{i=1}^{n} p_i \Rightarrow S_{last} \leq \frac{1}{m}(\sum_{i=1}^{n} p_i - p_{last})$



$C = S_{last} + p_{last}$

Makespan (P | | C_{max}): ρ=f(k)

$$C = s_{last} + p_{last} \leq \frac{1}{m}\sum_{i=1}^{n} p_i + (1 - \frac{1}{m}) p_{last}$$

$$\Rightarrow \mathrm{C} \leq \mathrm{C}^* + \frac{m-1}{m} p_{last}$$

$$\Rightarrow \mathrm{C} \leq \mathrm{C}^* + \frac{m-1}{m} p_{k+1}$$

$$\Rightarrow \frac{C}{\mathrm{C}^*} \leq 1 + \frac{(1-1/m)p_{k+1}}{\mathrm{C}^*}$$

Makespan ($P \mid \mid C_{max}$): ρ=f(k)

$$\frac{C}{\mathrm{C}^*} \leq 1 + \frac{(1-1/m)p_{k+1}}{\mathrm{C}^*}$$

Consider the first k+1 jobs: $p_i \geq p_{k+1}$

At least one processor executes $\left\lfloor \frac{k}{m} \right\rfloor + 1$ of them

Hence,

$$\mathrm{C}^* \geq (1+\lfloor k/m \rfloor)p_{k+1} \Rightarrow \frac{1}{\mathrm{C}^*} \leq \frac{1}{(1+\lfloor k/m \rfloor)p_{k+1}}$$

And thus, $$\frac{C}{\mathrm{C}^*} \leq 1 + \frac{(1-1/m)p_{k+1}}{(1+\lfloor k/m \rfloor)p_{k+1}} = 1 + \frac{1-1/m}{1+\lfloor k/m \rfloor}$$

Makespan (P | | C_{max}): ρ=f(k)

Recall the definition of PTAS and FPTAS:

- Polynomial Time Approximation Schemes (PTAS)
  - $C/C^* \leq 1 + \varepsilon$, for any $\varepsilon > 0$
  - Complexity: O(poly (|I|))
  - dependence on ε: allowed to be O(exp (1/ε)), e.g. $O(n^{3/\varepsilon})$

- Fully Polynomial Time Approximation Schemes (FPTAS)
  - $C/C^* \leq 1 + \varepsilon$, for any $\varepsilon > 0$
  - Complexity: O(poly (|I|))
  - Dependence on ε: O(poly (1/ ε)), e.g., $O((1/\varepsilon)^2 n^3)$

Makespan ($P \mid \mid C_{max}$): A PTAS

A PTAS from our new algorithm:

1. Given ε > 0, choose k such that $\frac{1-\frac{1}{m}}{1+\left\lfloor \frac{k}{m} \right\rfloor} \le \varepsilon \quad \left(\Rightarrow k > \frac{m-1}{\varepsilon} - m \right)$
2. Schedule optimally the k longest jobs
3. Schedule the rest of the jobs using LPT

$$\frac{C}{C^*} \le 1+\varepsilon, \qquad \forall \varepsilon > 0$$

Makespan (P | | C_{max}): A PTAS

Complexity

Step 1. O(1)

Step 2. O(m^k) (why ?)

Step 3. O(nlogn) (why ?)

Total: $O(n\log n + m^k) \sim O(n\log n + m^{\frac{m-1}{\varepsilon}-m})$

→ O(poly(n))

→ O(poly(m)) for FIXED m (when m=O(1))

→ O(exp(1/ε))

A PTAS for constant m

- There is also an FPTAS for fixed m
- And there is a PTAS for P| | C_{max} for arbitrary m [Hochbaum, Shmoys ’87]
- There is no FPTAS, for P| | C_{max}, for general m, unless P = NP

C. Knapsack problems

Knapsack problems

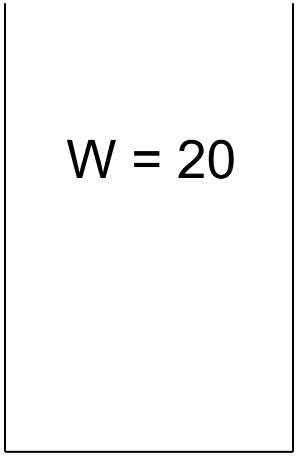
- We are given a knapsack with maximum capacity W, and a set $S = \{1,2,\ldots,n\}$ of n items

- Each item i has a weight w_i and a value v_i
  - assume all w_i, v_i and W are integers

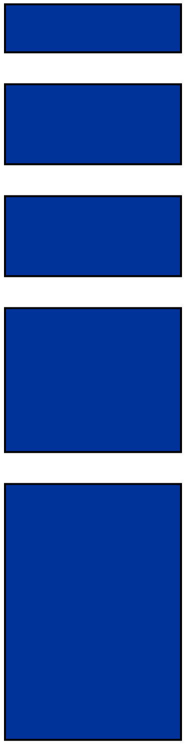
Problem: How to pack the knapsack to achieve maximum total value of packed items?

Knapsack problems

Max weight: W = 20



Items



Weight w_i	Value v_i
2	3
3	4
4	5
5	8
9	10

Knapsack problems

Three (basic) versions of the problem:

1. Fractional knapsack

   Items are divisible: any fraction of an item can go into the knapsack

   poly-time solvable by a greedy algorithm

2. 0-1 knapsack

   Items are indivisible: either take an item or not

   NP-complete, O(nW), by a dynamic programming algorithm, PTAS based on DP

3. Integer knapsack

   Multiple copies of indivisible items: take any number of copies of an item

   Exercise !

Knapsack problems

Fractional knapsack

$$\max \sum_{i \in S} v_i \, x_i, \text{ s.t. } \sum_{i \in S} w_i x_i \le W, \text{ and } x_i \in [0,1]$$

0-1 knapsack

$$\max \sum_{i \in S} v_i \, x_i, \text{ s.t. } \sum_{i \in S} w_i x_i \le W, \text{ and } x_i \in \{0,1\}$$

Integer knapsack

$$\max \sum_{i \in S} v_i \, x_i, \text{ s.t. } \sum_{i \in S} w_i x_i \le W, \text{ and } x_i \in N$$

Fractional Knapsack

Greedy algorithm:

- Start with an empty knapsack
- Consider the item with the maximum value per unit (v_i/w_i) among the remaining items,
- Take as much quantity of this item as the capacity of the knapsack allows

Note: at the end of the algorithm, the knapsack is loaded by the whole weights of all chosen items, except possibly the last included item

Theorem. Greedy algorithm computes an optimal solution for Greedy Knapsack in time O(nlogn).

Fractional Knapsack

Example:

Item	Weight	Value
1	10	60
2	20	100
3	30	120

Suppose W=50

The greedy algorithm will select:

- All of item 1
- All of item 2
- 2/3 of item 3

0-1 Knapsack

Brute-force approach

- there are 2^n possible combinations of n items
- Go through all combinations and find the one with the most total value and with total weight less or equal to W
- Running time: $O(n2^n)$

Can we do better?

  - Yes, by Dynamic Programming

DP for 0-1 Knapsack

Subproblem:

V[k, w] = maximum value of the subproblem consisting of the first k items $S_k=\{1, 2, \dots, k\}$ and capacity w (where $0 \le w \le W$)

Item k can either be in the optimal solution of V[k,w] or not

- First case: $w_k > w$
  - item k cannot be in the optimal solution of V[k, w]
  - $V[k,w] = V[k-1, w]$
- Second case: $w_k \le w$, thus item k could be in the optimal solution:
  - The maximum value of the subproblem consisting of the first k items and capacity w is one of the next two:
    - $V[k-1, w]$ or
    - the optimal solution of the subproblem consisting of the first k-1 items and capacity $w-w_k$, plus the value of item k: $V[k-1, w-w_k] + v_k$
  - $V[k,w] = \max \{ V[k-1, w], V[k-1, w-w_k] + v_k \}$

DP for 0-1 Knapsack

Recursive Formula

$$V[k,w] = \begin{cases} 0 & \text{if } k = 0 \text{ or } w = 0 \\ V[k-1,w] & \text{if } k,w \geq 1 \text{ and } w_k > w \\ \max\{V[k-1,w], V[k-1,w-w_k]+v_k\} & \text{if } k,w \geq 1 \text{ and } w_k \leq w \end{cases}$$

What we want is OPT = V[n, W]

DP for 0-1 Knapsack

```
0-1 Knapsack Value-only({$w_i$}, {$v_i$}, W)

for w := 0 to W do  V[0,w] := 0;

for k = 1 to n do
   for w := 0 to W do
      if $w_k \le w$                  // item i can be in the solution
         then  V[k,w]:= max { $v_k$ + V[k-1,w-$w_k$], V[k-1,w] }
         else  V[k,w]:= V[k-1,w]                 // $w_k > w$
```

Complexity: **O(n W)**

- Pseudopolynomial time algorithm
- Polynomial when W is small

Note: Almost all known pseudopolynomial time algorithms for NP-hard problems are based on dynamic programming

DP for 0-1 Knapsack

Can we find the actual set of items included in the optimal solution?

```
0-1 Knapsack($\{w_i\}$, $\{v_i\}$, W)
{
Run 0-1 Knapsack Value-only algorithm
k:=n; w:=W; S:= {}
While $k \neq 0$ and $w \neq 0$ do
   { if $V[k,w] \neq V[k-1,w]$ then  { S:= S U {k}; w:= w-$w_k$}
     k:=k-1 }
return S
}
```

complexity of this algorithm ?

Another DP for 0-1 Knapsack

Previous Algorithm:

OPT is equal to V[n,W] and can be found in O(nW) time

Another Dynamic Programming Algorithm:

Subproblem:

C[k,v] = the minimum capacity achieved when using only the items 1, 2,…, k, yielding a value equal to v.

OPT = maximum v for which C[n,v] $\leq$ W

Claim: The optimal solution can be found in $O(n^2 v_{max})$ time, where $v_{max} = \max_i v_i$

FPTAS for 0-1 KNAPSACK

- We will utilize the $O(n^2\ v_{max})$ dynamic programming algorithm for 0-1 KNAPSACK
- Recall that $v_{max} = \max_i \{v_i\}$ and $v_{max} \leq OPT \leq nv_{max}$
- Recall also that we have assumed all quantities are integers (the w_i's, the v_i's, and W)

FPTAS for 0-1 KNAPSACK

Main ingredients for designing an FPTAS

- Inspired by the dynamic programming algorithm
- The DP algorithm implies that if the values are small (polynomial in n), then we can solve the problem efficiently
- Idea: round down the values by ignoring some of their least significant bits
- Solve the "rounded" or "scaled" instance (which can be seen as a "perturbation" of the original instance)
- The scaling should be dependent on ε
- The scaled values should be bounded by a polynomial in n and 1/ε
- Prove that the solution found is a good approximation to the original instance

FPTAS for 0-1 KNAPSACK

SCALED INSTANCE:

Scale all item values by a parameter k, i.e., for item j, $v_j(k) = \lfloor v_j / k \rfloor$:

It holds that:

$$\frac{v_j}{k} - 1 < v_j(k) \quad (1) \qquad\qquad v_j(k) \le \frac{v_j}{k} \quad (2)$$

FPTAS-Knapsack(k)

```
{ Produce the scaled instance;
  Solve the scaled problem by the last DP algorithm;
  Let S(k) ⊆ {1,2,…,n} be the optimal solution to the scaled problem;
  Return S(k) for the original problem; }
```

The parameter k will be determined by the analysis

Theorem: The algorithm above with $k = \varepsilon v_{max}/n$ is an FPTAS for 0-1 KNAPSACK

FPTAS for 0-1 KNAPSACK

Proof:

- Let S* $\subseteq$ {1,2,…,n} be the optimal solution of original problem, with value OPT
- Let S(k) $\subseteq$ {1,2,…,n} be the output of the algorithm, with value OPT(k) for the original problem

(*) for the scaled instance, S(k) is of greater value than any other solution (hence better than S* too)

$$\boxed{\frac{v_j}{k} - 1 \overset{(1)}{<} v_j(k) = \left\lfloor \frac{v_j}{k} \right\rfloor \overset{(2)}{\leq} \frac{v_j}{k}}$$

$$OPT(k) = \sum_{j \in S(k)} v_j \overset{\text{by (2)}}{\geq} \sum_{j \in S(k)} k\ v_j(k)$$

$$= k \sum_{j \in S(k)} v_j(k) \overset{\text{by (*)}}{\geq} k \sum_{j \in S^*} v_j(k)$$

$$\overset{\text{by (1)}}{\geq} k \sum_{j \in S^*} \left(\frac{v_j}{k} - 1\right) = \sum_{j \in S^*} v_j - k \sum_{j \in S^*} 1 = OPT - k\left|S^*\right|$$

$$\geq OPT - kn, \quad \text{since} \quad \left|S^*\right| \leq n$$

FPTAS for 0-1 KNAPSACK

Proof (cont.):

Thus:

$$OPT(k) \geq OPT - n \cdot k,$$

$$\text{Choose} \quad k = \frac{\varepsilon \cdot v_{\max}}{n} \leq \frac{\varepsilon \cdot OPT}{n}, \text{since } v_{\max} \leq OPT$$

$$\text{Hence, } OPT(k) \geq OPT - \varepsilon \cdot OPT = (1-\varepsilon)OPT$$

Complexity

$$O\left(n^2 \left\lfloor \frac{v_{\max}}{k} \right\rfloor\right), \text{that is} \quad O(n^3 \frac{1}{\varepsilon}), \quad \text{since} \left\lfloor \frac{v_{\max}}{k} \right\rfloor = \frac{n}{\varepsilon}$$

O(poly(n, 1/ε)), hence FPTAS !

D. BIN PACKING

Bin Packing

Recall the problem:

BIN PACKING

I: A set of objects S = {1,…,n}, each with a positive integer weight w_i, i = 1,…,n, and a positive integer W (bin capacity)

Q: find a partition of S into $A_1, \dots, A_m$ (m bins) s.t. $\sum_{i \in A_j} w_i \leq W, j = 1,2,\dots,m$

and m is minimized

i.e., minimize the number of bins to fit the objects

Negative known-results:

- There is no approximation algorithm achieving a factor better than 3/2, unless P = NP
- The proof (see next slide) is by showing that it is NP-hard to distinguish between instances with OPT = 2 and instances with OPT = 3

Bin Packing

Unless $P \neq NP$, there is no $\left(\frac{3}{2} - \delta\right)$ -approximation algorithm for BIN-PACKING

Proof:

Assume that there is an algorithm A such that $m \leq \left(\frac{3}{2} - \delta\right) OPT$.

Run A for $M = \frac{1}{2}\sum_{i \in S} w_i$

If m=2 then PARTITION has answer YES!

If m $\geq$ 3 we have

$$m < \frac{3}{2} OPT \Rightarrow OPT > \frac{2}{3} m \geq \frac{2}{3} \cdot 3 = 2$$

So OPT>2 and thus, PARTITION has answer NO!

Hence, we have an O(poly) algorithm for PARTITION,

That is P=NP, a contradiction.

What if OPT increases with n?

Bin Packing

On the positive side:

Greedy algorithms can achieve constant approximations

First-Fit algorithm:

- Start with one empty bin
- Process the items in an arbitrary order
- Try to place the next item in one of the existing bins (if it fits)
- If not, then create a new bin and put it there

Theorem: First-Fit achieves a 2-approximation
Relatively simple (do it as an exercise)

Bin Packing

Improving on First-Fit:

First-Fit Decreasing algorithm (FFD):

- First sort the items in decreasing order
- Run First-Fit but by processing the items in this order

Theorem: FFD uses at most 11/9 OPT + 1 bins

Bin Packing

And further improvements:

- Bin Packing does not admit a PTAS (since we have 3/2-hardness result)
- But it does achieve an *Asymptotic* PTAS

Theorem [Fernandez de la Vega, Lueker, ’81]:
For any ε>0, there exists an algorithm using at most
(1 + ε)OPT + 1 bins

- Asymptotic refers to the fact that when OPT grows the approximation ratio approaches 1