ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Special Topics on Algorithms

Fall 2023

Introduction

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Special Topics on Algorithms

- A continuation of the Algorithms course
- Emphasis on topics not covered during the Algorithms course and also on some more modern topics and applications
- You can take this course during your 3rd year or later
- Prerequisites:
  - You have passed the Algorithms course
  - You liked the Algorithms course

Content – Topics to be covered

- Introduction
  - Some basic concepts
  - Distinction between polynomial, pseudopolynomial and exponential time algorithms
- Problems on numbers
  - Exponentiation/Fibonacci/Euclid’s Algorithm for GCD
  - Modular arithmetic, prime numbers, primality testing
  - Applications: public key cryptosystems, RSA and digital signatures

Content – Topics to be covered

- Average case analysis
  - Sorting: Insertionsort, Quicksort
  - Binary Search Trees, hashing
- Coping with NP-completeness – Approximation algorithms
  - Greedy and other combinatorial algorithms
    - Vertex Cover, Set Cover, Maximum Coverage, TSP
    - Partition, Knapsack, Scheduling, Bin Packing
    - SAT
- Randomized Algorithms
  - Max Cut, Min Cut, Max k-SAT

Content – Topics to be covered

- Flows and Matchings
  - Fundamental algorithms for the Maximum Flow in a network graph and the Maximum Matching in bipartite graphs.
- (Integer) Linear Programming
  - Applications and LP based Approximation Algorithms
  - LP duality
- Invited lectures
  - We may have 2 lectures by other faculty members and collaborators on some applications

Bibliography

- [DPV] S. Dasgupta, C. H. Papadimitriou, U. V. Vazirani : "Algorithms"
- [CLRS] T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein: "Introduction to Algorithms"
- [KT] J. Kleinberg, E. Tardos: "Algorithm Design"
- …
- **and many resources on the WWW**

Communication

- Office hours:
  - Tuesdays: 12:00 – 14:00
  - Fridays: 13:00 – 14:00

- You can always email me regarding questions
  - If I do not reply within 3 days, send it again

- Eclass: Ειδικά Θέματα Αλγορίθμων
  - Please check the announcements there at least once per week

Tutorials

- Teaching Assistant: Panagiotis Tsamopoulos
- Office hours for the TA to be announced soon
- Tutorials starting next week

Grading

Final exam	**65%**
Midterm exam	**20%**
Individual Assignments (x2)	**15%**

Note: The midterm is used only if it helps your final grade, otherwise the final exam will count as 85%

Date of midterm: towards end of November

Introductory concepts:
Polynomial, Pseudo-Polynomial and Exponential Algorithms

What are we interested in?

Problems to be solved by a machine: precisely defined; no ambiguities

- We want to transform appropriately the input data (problem instances) to output data
- Two subcategories are decision and optimization problems.

COMPUTATIONAL PROBLEM

A problem where we are given **input** instances and some computational question and we want to find an answer/**output**:
E.g., given a graph we wish to compute the set of vertices of odd degree, or to compute a set of k vertices where every pair of them is connected by an edge.

Examples of Problems

EXP(onentiation)

I: positive integers a,n
Q: calculate a^n

FIBONACCI NUMBERS

I: a positive integer n
Q: calculate the n-th Fibonacci number F_n

SUBSET SUM

I: a set $S=\{a_1, a_2, \dots, a_n\}$ of n positive integers and an integer B
Q: is there a subset $A \subseteq S$ s. t. $\sum_{i \in A} a_i = B$?

SAT(isfiability)

I: a boolean formula ϕ
Q: Is ϕ satisfiable ?

(is there a value assignment to its variables making ϕ TRUE ?
= truth assignment)

Algorithms

Three crucial questions about any algorithm for any problem:

1. **Is it correct ?**
   - Does it always terminate?
   - Does it give a correct answer for any instance of the problem ?
2. **How much time/space does it take, as a function of its input?**
   - "time" = number of steps / "space" = number of bits in memory
   - "time" independent of language/implementation/machine
   - We mostly focus on time, expressed as a function $T(n)$, where n is the size of the instance we try to solve
   - Interested in asymptotic behavior of $T(n)$
   - Notation: O, Ω, Θ, o, ω
3. **Can we do better ?**

Time Complexity of an algorithm

There are many instances of the same size

How does the algorithm work over all these instances?

Best-case complexity

- The minimum number of steps taken on any instance of size n
- Not useful, too optimistic

Worst-case complexity

- The maximum number of steps taken on any instance of size n
- An upper bound on the complexity of the problem
- The most usual analysis

Average case complexity

- The average number of steps taken on any instance of size n
- Depends on the distribution of instances (use of probabilities)

Time Complexity of a problem and lower bounds

Complexity of a problem Π: $T_\Pi(n)$

The (worst case) complexity of the best (known) algorithm A

$$T_\Pi(n) = \min_A \{T_A(n)\}$$

Obtaining a lower bound on a problem's complexity $L_\Pi(n)$:

- By proving that there is no algorithm with $T_A(n) < L_\Pi(n)$
- Rare results (e.g., log(n!) for sorting).

Optimal algorithm

- An algorithm A, for which $T_A(n) = L_\Pi(n)$
- For many problems we still do not know if we have found an optimal algorithm
- Even for well-studied problems, new improvements arise over the years

Algorithm Analysis

- Proof of correctness
  - Some times for a well defined subset of input instances
- Evaluation of time complexity
  - Average, worst, best case
- Appropriate solution depending on the application requirements

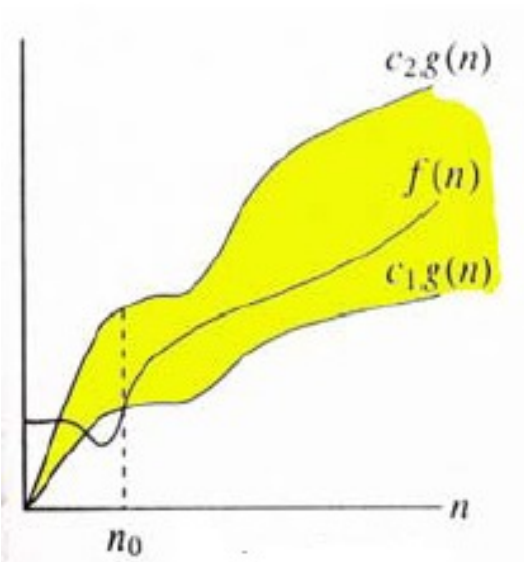
Benefits of theoretical analysis:

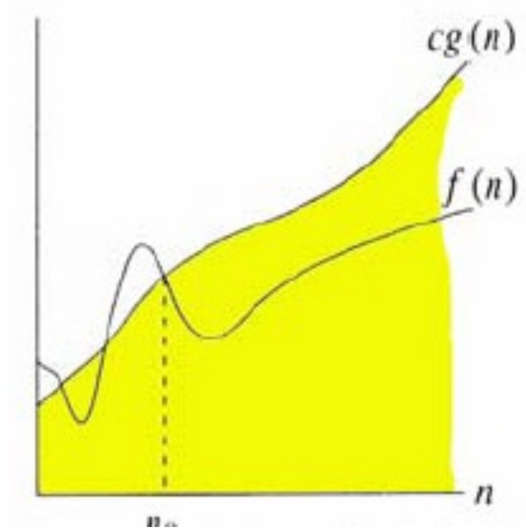
- Do not require experimental evaluation but only concrete description of the algorithm
- Results into general conclusions easy to verify, by considering all input instances, determining the time complexity as function of the input size

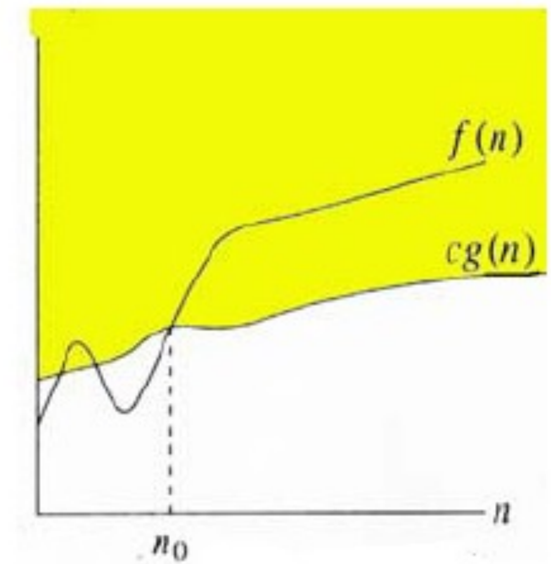
Mathematical background: discrete math (graphs, recurrence relations, combinatorics), mathematical logic, induction in all its forms (simple, strong, structural)

Asymptotic Notation

In pictures:







$f(n) = \Theta(g(n))$ $\quad$ $f(n) = O(g(n))$ $\quad$ $f(n) = \Omega(g(n))$

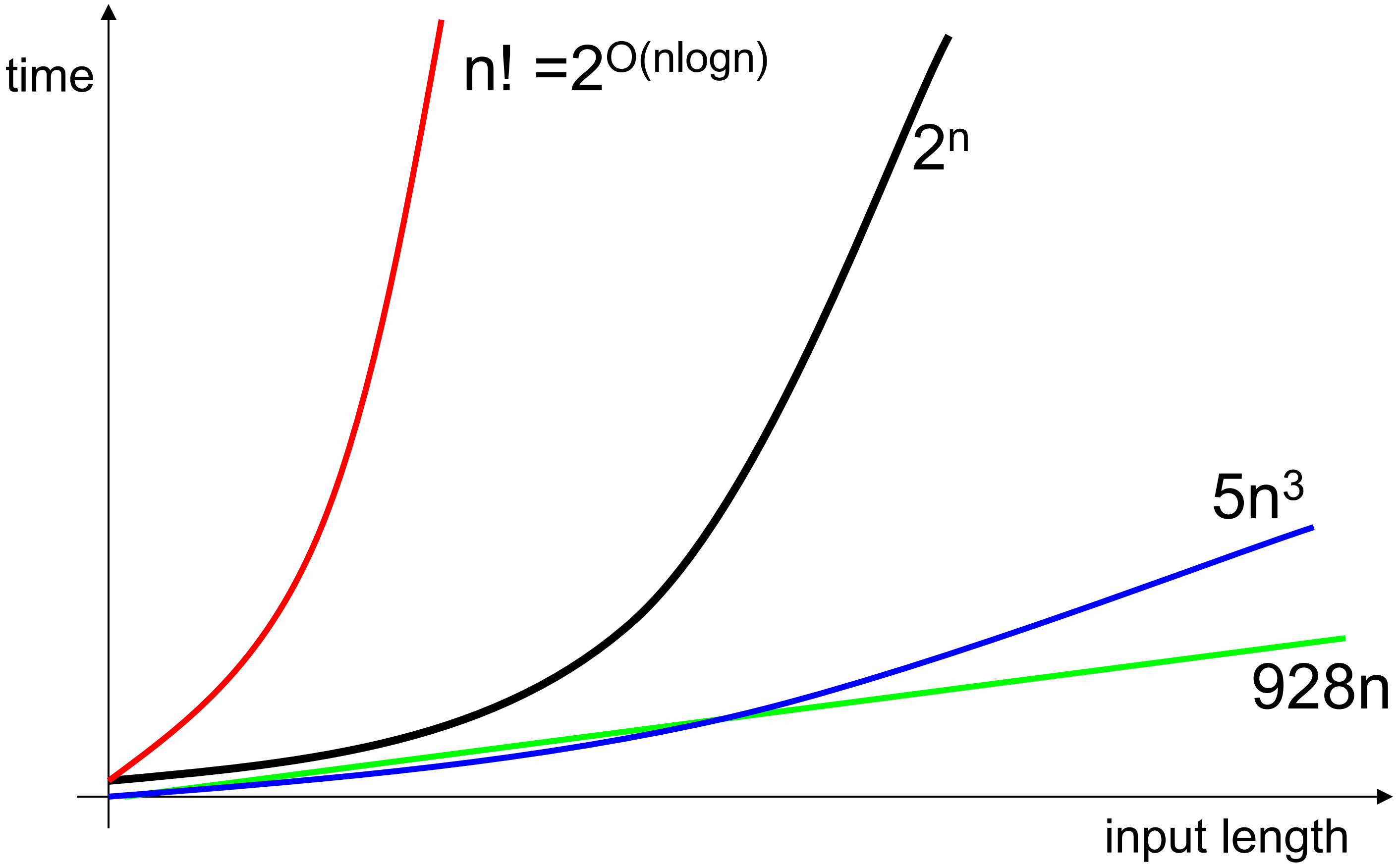
Asymptotic Notation

More formally:

- A function $f(n)$ is $O(g(n))$ if there exist positive constants c_0 and n_0 such that $f(n) \leq c_0 g(n)$ for every $n \geq n_0$
  - The constant c_0 might be large (but still constant, **independent of** n)
  - Examples:
    - $2n + 10$ is $O(n)$. It suffices to set $c_0 = 3$ and $n_0 = 10$
    - $4n\log n + 150n + 3000\text{sqrt}(\log n) = O(n\log n)$. Set $c_0 = 3154$, $n_0 = 1$

- A function $f(n)$ is $\Omega(g(n))$ if there exist positive constants c_0 and n_0 such that $f(n) \geq c_0 g(n)$ for every $n \geq n_0$
- A function $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$

Growth of various functions



Size of instance and complexity

Consider the description of an instance (i.e., of all the parameters and constraints)

|I| = length of encoded instance/input

Instance $\xrightarrow[\text{e.g. in decimal / binary / unary}]{\text{encoding}}$ encoded instance I

|I| = # of digits of the encoded input

Integer n:	Decimal	Binary	Unary
# bits :	$\lfloor \log_{10} n \rfloor + 1$	$\lfloor \log_2 n \rfloor + 1$	n

Size of instance and complexity

- We typically use the binary encoding
  - but there are reasons to consider other encodings too in complexity theory
- Hence, unless otherwise stated, |I| = # of bits of the encoded input
- Let also N(I) = the largest number in the input
  - Applicable only for problems that have numeric parameters in their input, like Knapsack
- Classification of algorithms
  - Polynomial algorithms: running time O(poly(|I|)
  - Exponential algorithms: running time Θ(exp(|I|)
  - Pseudo-Polynomial algorithms: Θ(poly(N(I)), which in worst case is Θ(exp(|I|)
    - We can say that they are O(poly(|I|)) <u>if we consider I encoded in unary !</u> (i.e, polynomial when N(I) not too large)
    - Example: Knapsack admits a dynamic programming algorithm with running time $O(n^2\ v_{max})$, where v_{max} is max value in the instance
    - Only relevant for problems with numeric parameters!
    - Not relevant for SAT

Recap from the Algorithms course: Analyzing Recurrence Relations

The Master Theorem

- How do we analyze recurrence relations?
- There are various methods
- The substitution method:
  - Keep substituting until you guess the solution
  - Use induction to prove it formally

Example: $T(n) = T(n-1) + n$, $T(1) = 1$

- $T(n) = T(n-1) + n$
- $= (T(n-2) + n-1) + n$
- $= T(n-2) + n + n-1$
- $= (T(n-3) + n-2) + n + n-1$
- $= \ldots$
- $= n + n-1 + n-2 + \ldots + 2 + 1 = O(n^2)$

Is there a general result that could be applicable to the recurrence relations we will encounter?

The Master Theorem

If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ for some constants $a > 0$, $b > 1$, $d \geq 0$, then

$$T(n) = \begin{cases} \Theta(n^d), & \text{if } d > \log_b a \quad (b^d > a) \\ \Theta(n^d \log_b n), & \text{if } d = \log_b a \quad (b^d = a) \\ \Theta(n^{\log_b a}), & \text{if } d < \log_b a \quad (b^d < a) \end{cases}$$

- Usually convenient to think of n as a power of b, so that n/b is an integer.
- In many cases of interest, b = 2
- More general versions of this theorem are available as well

The Master Theorem - Examples

- Naive integer multiplication (by divide and conquer)
  - $T(n) = 4T(n/2) + O(n)$
  - $a = 4$, $b = 2$, $\log_b a = \log_2 4 = 2$
  - $d = 1 < 2 = \log_b a$
  - Case (iii) applies: $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(\mathrm{n}^2)$
- Karatsuba's algorithm for integer multiplication
  - $T(n) = 3T(n/2) + O(n)$
  - $a = 3$, $b = 2$, $\log_b a = \log_2 3 = 1.59$
  - $d = 1 < \log_b a$
  - Case (iii) applies again: $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(\mathrm{n}^{1.59})$

The Master Theorem - Examples

- $T(n) = 5T(n/25) + O(n^2)$
  - $a = 5$, $b = 25$, $\log_b a = \log_{25} 5 = 0.5$
  - $d = 2 > 0.5 = \log_b a$
  - case (i) applies: $T(n) = \Theta\left(n^d\right) = \Theta(\mathrm{n}^2)$
- $T(n) = T(2n/3) + O(1)$
  - $a = 1$, $b = 3/2$, $\log_b a = \log_{3/2} 1 = 0$
  - $d = 0 = \log_b a$
  - case (ii) applies: $T(n) = \Theta\left(n^0 \log_{3/2} n\right) = \Theta(\mathrm{logn})$
- $T(n) = 9T(n/3) + O(n)$
  - $a = 9$, $b = 3$, $\log_b a = \log_3 9 = 2$
  - $d = 1 < 2 = \log_b a$ $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(\mathrm{n}^2)$
  - case (iii) applies: