Mathematical Background

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Some Mathematical Tools we Need

- In the next slides we are summarizing some important properties of vector and affine spaces in order to:
  - Establish a formal representation of our data and their operations
  - Provide the mathematical tools to process and extract information from our geometrical representations

Vector Spaces (1)

- A set *V* with elements called *vectors* and denoted $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{v}}$ etc. is a vector space if two operations are defined:
  - *vector addition* between two vectors, denoted $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ whose result is also a vector
  - *scalar multiplication* between a scalar and a vector denoted $\lambda\vec{\mathbf{a}}$, whose result is also a vector
- and the following properties are satisfied:

Vector Spaces (2)

- Addition properties:
  - *Commutativity*: $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}, \ \forall\, \vec{\mathbf{a}}, \vec{\mathbf{b}} \in V$
  - *Associativity*: $\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}, \ \forall\, \vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}} \in V$
  - Existence of a *zero element* $\overrightarrow{\mathbf{0}} \in V$: $\vec{\mathbf{a}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{a}} = \vec{\mathbf{a}}$, $\forall\, \vec{\mathbf{a}} \in V$
  - *Inversibility:* $\forall\, \vec{\mathbf{a}} \in V, \exists\, \vec{\mathbf{a}}' = -\vec{\mathbf{a}}$: $\quad \vec{\mathbf{a}} + (-\vec{\mathbf{a}}) = \overrightarrow{\mathbf{0}}$

Vector Spaces (3)

- Scalar multiplication properties:
  - *Associativity*: $\lambda(\mu\vec{\mathbf{a}}) = (\lambda\mu)\vec{\mathbf{a}}$, $\forall\, \vec{\mathbf{a}} \in V$ and $\forall\, \lambda, \mu \in \mathbb{R}$
  - *Identity element*: $\overrightarrow{1 \cdot \mathbf{a}} = \vec{\mathbf{a}}$, $\forall\, \vec{\mathbf{a}} \in V$
  - *Distributivity* of scalar multiplication over vector addition: $\lambda(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \lambda\vec{\mathbf{a}} + \lambda\vec{\mathbf{b}}$, $\forall\, \vec{\mathbf{a}}, \vec{\mathbf{b}} \in V$ and $\forall\, \lambda \in \mathbb{R}$
  - *Distributivity* of vector addition over scalar multiplication: $(\lambda + \mu)\vec{\mathbf{a}} = \lambda\vec{\mathbf{a}} + \mu\vec{\mathbf{a}}$, $\forall\, \vec{\mathbf{a}} \in V$ and $\forall\, \lambda, \mu \in \mathbb{R}$

2D and 3D Vectors

- The common 2D and 3D vectors we use in computer graphics form corresponding vector spaces
- For 3D:

$$\vec{\mathbf{v}} = (x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x \quad y \quad z]^T$$

- With the following well-known operations:

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = [a_x + b_x \quad a_y + b_y \quad a_z + b_z]^T$$

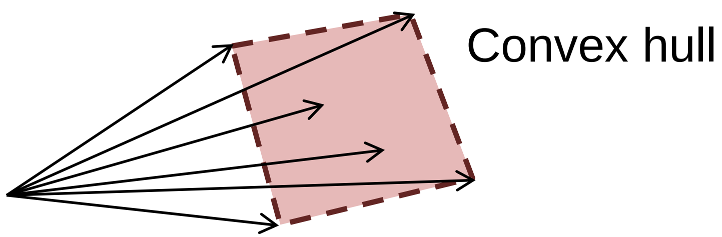
$$\lambda\vec{\mathbf{a}} = [\lambda a_x \quad \lambda a_y \quad \lambda a_z]^T$$

Linear Combinations

- For a set of vectors $\overrightarrow{\mathbf{a}_1}, \overrightarrow{\mathbf{a}_2}, \dots, \overrightarrow{\mathbf{a}_k} \in V$, an expression of the form:

$\vec{\mathbf{v}} = \lambda_1 \overrightarrow{\mathbf{a}_1} + \lambda_2 \overrightarrow{\mathbf{a}_2} + \dots + \lambda_k \overrightarrow{\mathbf{a}_k}$, $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$ is a linear combination of these vectors.

- If $\sum_{i=1}^{k} \lambda_i = 1$, then this is an affine combination
- If additionally, $\lambda_1, \lambda_2, \dots, \lambda_k \geq 0$, it is a convex combination, and we say that $\vec{\mathbf{v}}$ resides within the convex hull of $\overrightarrow{\mathbf{a}_1}, \overrightarrow{\mathbf{a}_2}, \dots, \overrightarrow{\mathbf{a}_k}$



Linear Independence

- $\overrightarrow{\mathbf{a}_1}, \overrightarrow{\mathbf{a}_2}, \dots, \overrightarrow{\mathbf{a}_k} \in V$ are linearly independent if: $\vec{\mathbf{0}} = \lambda_1 \overrightarrow{\mathbf{a}_1} + \lambda_2 \overrightarrow{\mathbf{a}_2} + \dots + \lambda_k \overrightarrow{\mathbf{a}_k}$ only when:
  $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$
- Direct consequence:
  - If a vector can be written as a linear combination of some linearly independent vectors $\overrightarrow{\mathbf{a}_1}, \overrightarrow{\mathbf{a}_2}, \dots, \overrightarrow{\mathbf{a}_k}$, this expression is *unique*

Basis of a Vector Space

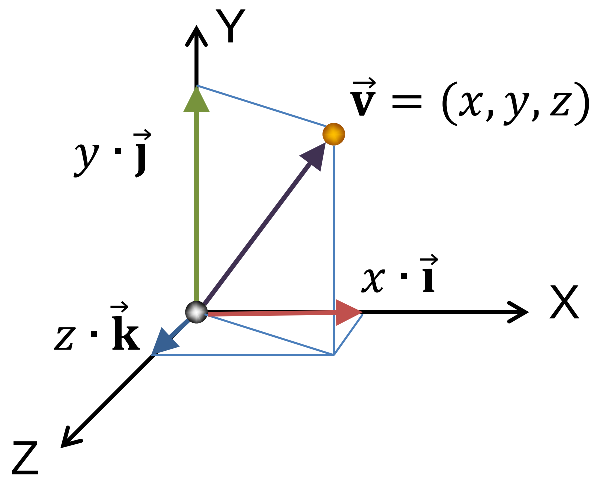
- A *basis* of a vector space is a set of linearly independent vectors having the additional property that every vector of the space can be written as a linear combination of them
- The (unique) coefficients with which a vector is written as a linear combination of the elements of a basis are called the *coordinates* of the vector in terms of this basis.
- Every vector space has at least one basis
- The number of elements in a vector space basis is called the *dimension* of the vector space.

Coordinates and Coordinate Systems

- In 3D we typically use the orthonormal basis:

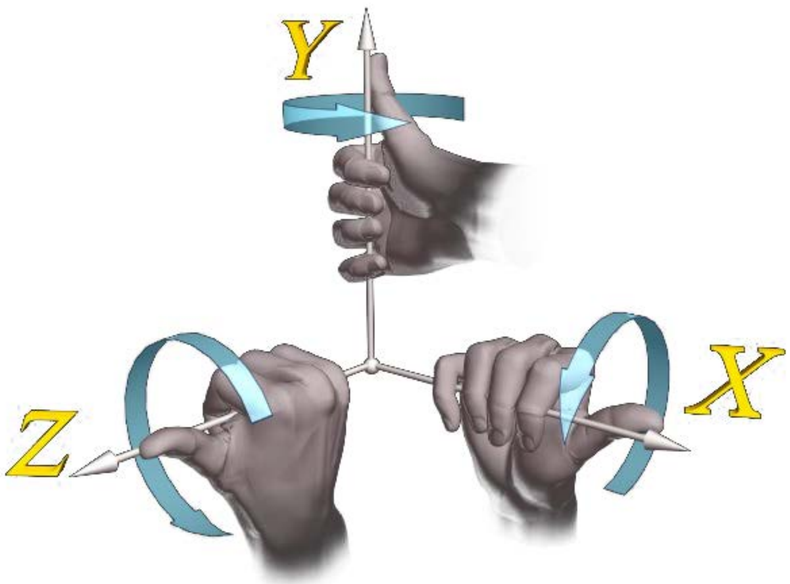
  $(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}})$

  $\vec{\mathbf{i}} = (1,0,0),\ \vec{\mathbf{j}} = (0,1,0), \vec{\mathbf{k}} = (0,0,1)$
- Similarly, we use $\vec{\mathbf{i}} = (1,0), \vec{\mathbf{j}} = (0,1)$ for 2D space

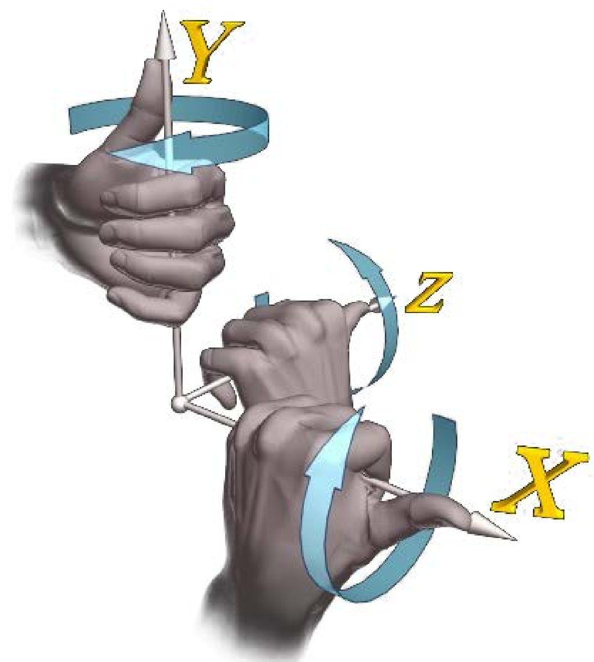


Coordinate System Conventions (1)

- In 3D space, we can use an arrangement of the axes so that the z axis points either “towards” us or “away” from us:



The “right-handed” (counter-clockwise) system



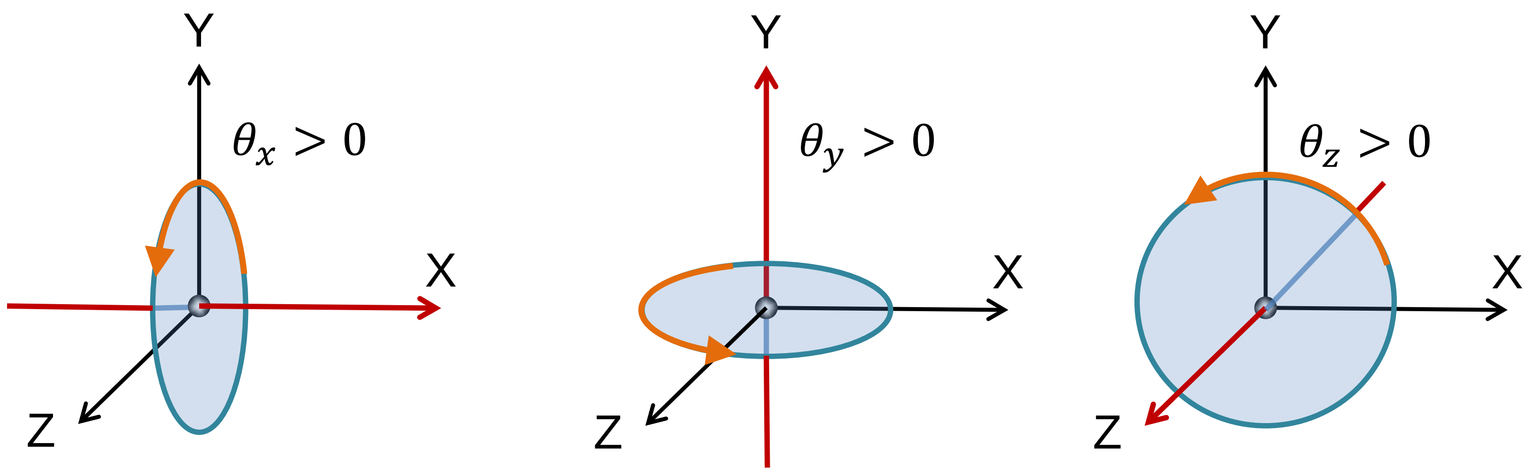
The “left-handed” (clockwise) system

Coordinate System Conventions (2)

- We most frequently use the right-handed (CCW) system in computer graphics (z axis pointing “outwards” to us, x pointing right, y up)

Coordinate System Conventions (3)

- Positive angles are counter-clockwise
  - Conveniently, we can use the rule of thumb (see previous slide) to determine the winding



Vector Norm

- The norm of a vector is a non-negative real number, which is actually the length of the vector:

$$|\vec{\mathbf{a}}| = \sqrt{x^2 + y^2 + z^2}$$

- Vectors with norm 1 are called unit vectors
- Given any vector with non-zero norm, we can obtain a corresponding unit vector via a process called normalization:

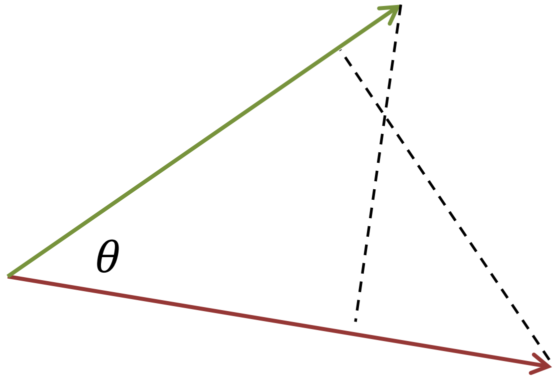
$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{1}{|\vec{\mathbf{a}}|}[x\ y\ z]^T$$

Dot (Inner) Product

- The dot product of two vectors is defined as:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$$

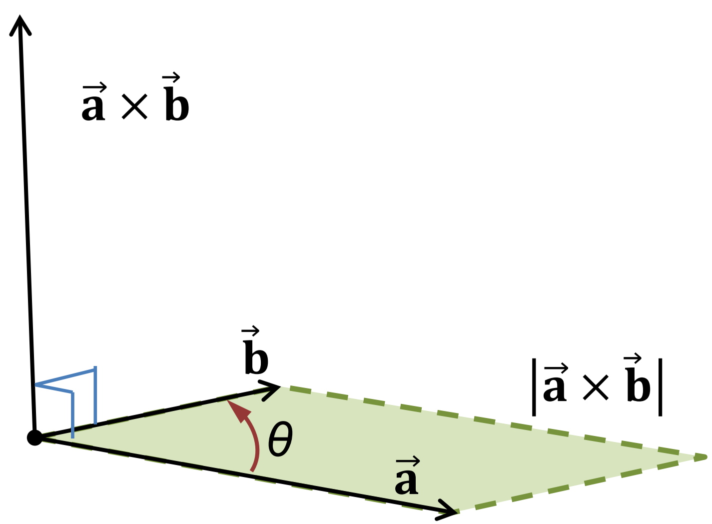
- Properties:
  - Commutativity: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$
  - Bilinearity: $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \lambda\vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \lambda(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})$
- The dot product is also:
- $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$:
  - $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0 \Leftrightarrow \vec{\mathbf{a}} \perp \vec{\mathbf{b}}$



Cross (External) Product

- The cross product of two 3D vectors is perpendicular to both of them and is defined as:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y, \quad a_z b_x - a_x b_z, \quad a_x b_y - a_y b_x)$$



Properties:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$$

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}})$$

$$\left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right| = |\vec{\mathbf{a}}|\left|\vec{\mathbf{b}}\right| \sin\theta$$

$\left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right|$ equals the area of the parallelogram $\vec{\mathbf{a}}, \vec{\mathbf{b}}$

Affine Spaces

- A set S of elements $\mathbf{p}, \mathbf{q}$, etc. called points is an affine space with an associated vector space V, if an operation called *addition* is defined between a point and a vector whose result is a point.
- Addition must obey the following properties:
  - *Associativity*: $(\mathbf{p} + \vec{\mathbf{a}}) + \vec{\mathbf{b}} = \mathbf{p} + (\vec{\mathbf{a}} + \vec{\mathbf{b}})$
  - *Zero element*: $\mathbf{p} + \vec{\mathbf{0}} = \mathbf{p}, \ \forall\, \mathbf{p} \in S$
  - Difference: $\forall\, \mathbf{p}, \mathbf{q} \in S, \exists\, \vec{\mathbf{a}} \in V: \mathbf{p} + \vec{\mathbf{a}} = \mathbf{q}$ and $\mathbf{q} - \mathbf{p} = \vec{\mathbf{a}}$
- In graphics, we use 2D and 3D points defined in the Euclidean spaces $\mathbb{E}^2, \mathbb{E}^3$

Points vs Vectors

- Affine spaces have no origin (no reference point) → we cannot inherently define coordinates, which requires a vector space! So:
  - Adding two points has no meaning
  - Using a point as a reference and adding a vector yields another point
  - The difference of two points constructs a vector
- Points denote position
- Vectors have direction and magnitude, but are not based on a specific point

Coordinate Systems for Points

- If we consider a specific point $\mathbf{o} \in \boldsymbol{S}$ as reference (i.e. an *origin*) and a basis $(\overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \dots \overrightarrow{\mathbf{b}_n})$ of the associated vector space V, then $(\mathbf{o}, \overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \dots \overrightarrow{\mathbf{b}_n})$ constitutes an (affine) coordinate system of *S*
- Given a point $\mathbf{p} \in \boldsymbol{S}$ so that:

$$\mathbf{p} - \mathbf{o} = \lambda_1 \overrightarrow{\mathbf{b}_1} + \lambda_2 \overrightarrow{\mathbf{b}_2} + \cdots + \lambda_n \overrightarrow{\mathbf{b}_n}$$

- $\lambda_1, \lambda_2, \dots, \lambda_n$ are the coordinates of $\mathbf{p}$ w.r.t $(\mathbf{o}, \overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \dots \overrightarrow{\mathbf{b}_n})$

Contributors

- Georgios Papaioannou

- Sources:
  - T. Theoharis, G. Papaioannou, N. Platis, N. M. Patrikalakis, Graphics & Visualization: Principles and Algorithms, CRC Press