Industrial Economics

TA Session 1 — Worked Solutions

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Session Overview

These notes contain complete, step-by-step solutions for every problem discussed in TA Session 1. Problems are drawn from two problem sets:

- **Exercise A** (PS I, Problem 4): Two-Period Monopoly with Learning-by-Doing
- **Exercise B** (PS I, Problem 7): Third-Degree Price Discrimination
- **Exercise C** (PS II, Problem 1): Cournot Oligopoly vs. Monopoly
- **Exercise D** (PS II, Problem 2): Cournot Oligopoly with Asymmetric Costs
- **Exercise E** (PS II, Problem 6): Bertrand Oligopoly with Asymmetric Costs
- **Exercise F** (PS II, Problem 7): Bertrand Oligopoly with Differentiated Products

For each exercise you will find the full algebraic derivation, tabular summaries where appropriate, economic interpretation of the results, and notes on the most common mistakes.

Contents

Exercise A · PS I, Problem 4 — Two-Period Monopoly with Learning-by-Doing

Problem Statement

Consider a monopolist that produces a single good. The monopolist sells the good in two consecutive periods, $t = 1$ and $t = 2$. The demand for the good is $q_t = 1 - p_t$. Marginal (and average) cost is equal to $c \in [0, 1)$ in $t = 1$ and to $c - \lambda q_1$ in $t = 2$, where $\lambda \in (0, 1)$.

(i) Based on the provided information, do you think that demands and costs of the two periods are dependent or independent?

(ii) Find the prices that the monopolist charges in each period (assume that the discount factor is 1).

(iii) Compute the Lerner index and the elasticity of demand in equilibrium for both periods. Discuss your findings.

Part (i) — Dependence of demands and costs

Demands are independent. The quantity sold in period t depends only on the price charged in that same period: $q_t = 1 - p_t$. There is no inter-temporal demand link (the good is not durable and consumers do not stockpile).

Costs are dependent. The period-2 marginal cost $\text{MC}_2 = c - \lambda q_1$ falls with the quantity produced in period 1. The more the firm produces today, the lower its costs tomorrow. This is the hallmark of *learning-by-doing*: accumulated production experience reduces future unit costs.

Part (ii) — Equilibrium prices

Setting up the joint optimisation problem. Let π_t denote the per-period profit in period t and Π the present value of total profit (discount factor = 1):

$$\pi_1 = (p_1 - c)\, q_1 = (p_1 - c)(1 - p_1)$$

$$\pi_2 = (p_2 - \text{MC}_2)\, q_2 = \Big[p_2 - c + \lambda(1 - p_1)\Big](1 - p_2)$$

$$\Pi = \pi_1 + \pi_2 = (p_1 - c)(1 - p_1) + \Big[p_2 - c + \lambda(1 - p_1)\Big](1 - p_2) \tag{1}$$

where we substituted $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. Because MC_2 depends on p_1 via $q_1 = 1 - p_1$, the monopolist must *jointly* maximise Π over (p_1, p_2).

First-order conditions (FOCs).

$$\frac{\partial \Pi}{\partial p_1} = \underbrace{(1-p_1)-(p_1-c)}_{\partial \pi_1/\partial p_1} + \underbrace{(-\lambda)(1-p_2)}_{\partial \pi_2/\partial p_1} = 0$$

$$\implies \quad 1 - 2p_1 + c - \lambda(1-p_2) = 0 \tag{2}$$

$$\frac{\partial \Pi}{\partial p_2} = (1-p_2) - \Big[p_2 - c + \lambda(1-p_1)\Big] = 0$$

$$\implies \quad 1 - 2p_2 + c - \lambda(1-p_1) = 0 \tag{3}$$

Common Mistake to Avoid

The term $-\lambda(1-p_2)$ in (2) is the key inter-temporal link. Raising p_1 lowers $q_1 = 1-p_1$, which raises $\text{MC}_2 = c - \lambda q_1$, which in turn reduces period-2 profit — hence the negative cross-period term $\partial\pi_2/\partial p_1 = -\lambda(1-p_2) < 0$. Students who maximise π_1 and π_2 *independently* will set $1-2p_1+c=0$, missing this term entirely and obtaining a price that is too high in period 1.

Solving the system. Subtracting (3) from (2):

$$\Big(-2p_1+\lambda p_2\Big) - \Big(-2p_2+\lambda p_1\Big) = 0 \implies (2+\lambda)(p_2-p_1) = 0$$

Since $2+\lambda>0$, we get $p_1^* = p_2^* \equiv p^*$. Substituting p^* for both prices in (3):

$$1-2p^*+c-\lambda(1-p^*) = 0 \implies p^*(2-\lambda) = 1-\lambda+c \implies p^* = \frac{1-\lambda+c}{2-\lambda}$$

Equilibrium quantity: $q^* = 1-p^* = \dfrac{1-c}{2-\lambda}$.

Key Result

$$p_1^* = p_2^* = \frac{1-\lambda+c}{2-\lambda}, \qquad q_1^* = q_2^* = \frac{1-c}{2-\lambda}$$

Comparison with static monopoly. A one-period monopolist maximises $\pi = (p-c)(1-p)$, giving $p^{\text{static}} = (1+c)/2$ and $q^{\text{static}} = (1-c)/2$. Since $\lambda \in (0,1)$:

$$q^* = \frac{1-c}{2-\lambda} > \frac{1-c}{2} = q^{\text{static}}$$

The learning-by-doing monopolist prices *lower* (sells *more*) than a static monopolist in each period, because it internalises the future cost savings that higher output generates.

Part (iii) — Lerner index and demand elasticity

Throughout, we define the price elasticity of demand as a positive number:

$$\eta = -\frac{\partial q}{\partial p}\cdot\frac{p}{q} > 0$$

For $q = 1-p$ this gives $\eta = -(-1)\cdot p/q = p/q > 0$. The standard Lerner condition then reads $L = 1/\eta$.

Period-2 marginal cost in equilibrium. Substituting q_1^* into $\text{MC}_2 = c - \lambda q_1$:

$$\text{MC}_2^* = c - \lambda \cdot \frac{1-c}{2-\lambda} = \frac{c(2-\lambda) - \lambda(1-c)}{2-\lambda} = \frac{2c-\lambda}{2-\lambda} \tag{4}$$

Price elasticity at the equilibrium. Both periods share the same demand curve and the same equilibrium (p^*, q^*), so η is the same in both periods:

$$\eta = \frac{p^*}{q^*} = \frac{(1-\lambda+c)/(2-\lambda)}{(1-c)/(2-\lambda)} = \frac{1-\lambda+c}{1-c} \qquad (\eta > 0) \tag{5}$$

Hence $1/\eta = (1-c)/(1-\lambda+c)$.

Lerner index, period 2.

$$L_2 = \frac{p^* - \text{MC}_2^*}{p^*} = \frac{\dfrac{1-\lambda+c}{2-\lambda} - \dfrac{2c-\lambda}{2-\lambda}}{\dfrac{1-\lambda+c}{2-\lambda}} = \frac{1-c}{1-\lambda+c} \tag{6}$$

Comparing with $1/\eta$:

$$\boxed{L_2 = \frac{1}{\eta}}$$

The standard Lerner condition holds in period 2.

Lerner index, period 1. Using the accounting marginal cost $\text{MC}_1 = c$:

$$L_1 = \frac{p^* - c}{p^*} = \frac{1-\lambda+c-c(2-\lambda)}{1-\lambda+c} = \frac{(1-\lambda)(1-c)}{1-\lambda+c} = \frac{1-\lambda}{\eta} \tag{7}$$

Since $\lambda \in (0,1)$, we have $0 < 1-\lambda < 1$ and therefore:

$$\boxed{L_1 = \frac{1-\lambda}{\eta} < \frac{1}{\eta} = L_2}$$

Key Result

$$L_2 = \frac{1}{\eta} = \frac{1-c}{1-\lambda+c} \qquad \text{but} \qquad L_1 = \frac{(1-\lambda)(1-c)}{1-\lambda+c} < \frac{1}{\eta}$$

Economic Intuition

The period-1 Lerner index is lower than $1/\eta$, meaning the firm appears to under-exploit its market power in period 1. This is not inefficiency — it is optimal forward-looking behaviour.
Every extra unit sold in period 1 reduces the period-2 marginal cost by λ. The true (shadow) cost of period-1 production is therefore not c, but $c - \lambda q_2^* = \text{MC}_2^*$. If we

recompute L_1 using MC_2^* as the effective cost:

$$\frac{p^* - \text{MC}_2^*}{p^*} = \frac{1-c}{1-\lambda+c} = \frac{1}{\eta}$$

The standard Lerner condition is restored. The firm is *investing* in lower future costs by deliberately setting a price below the static monopoly level in period 1.

Exercise B · PS I, Problem 7 — Third-Degree Price Discrimination

Problem Statement

Purple Dream has the monopoly on the production of purple light-emitting diodes (LEDs). It faces geographically separated markets, denoted A and B. The demands on these two markets are respectively given by $q_A = 1 - p_A$ and $q_B = \frac{1}{2} - p_B$. The transport and production costs are set to zero.

(i) Assume that the firm chooses to set a **uniform price** across the two markets. What is the profit-maximising uniform price? What are the quantities sold on the two markets at this price?

(ii) Assume that the firm uses **third-degree price discrimination**. What are the profit-maximising prices and quantities on the two markets?

(iii) Calculate consumer surplus and profit under a uniform price and under third-degree price discrimination. Compare these two situations and comment on the result.

(iv) Does the result that you have found in part (iii) hold generally? How would the results change if $q_B = \frac{1}{3} - p_B$?

Part (i) — Uniform pricing

Under uniform pricing, the firm sets a single price p in both markets. Demand is positive in both markets only if $p \leq \frac{1}{2}$ (the choke price of market B).

Case 1: $p \leq \frac{1}{2}$ (both markets served). Aggregate demand: $Q(p) = (1-p) + (\frac{1}{2} - p) = \frac{3}{2} - 2p$.
Profit: $\pi^U = p\left(\frac{3}{2} - 2p\right)$.
FOC: $\frac{3}{2} - 4p = 0 \implies p^U = \frac{3}{8}$.
This satisfies $p^U = \frac{3}{8} \leq \frac{1}{2}$, so the solution is interior.

Case 2: $p > \frac{1}{2}$ (market B excluded; only A served). Profit: $\pi = p(1-p)$, FOC: $1 - 2p = 0 \implies p = \frac{1}{2}$, yielding $\pi = \frac{1}{4} = \frac{8}{32}$.

Comparison. At $p^U = \frac{3}{8}$: $\pi^U = \frac{3}{8} \cdot \frac{3}{4} = \frac{9}{32} > \frac{8}{32}$. The interior solution dominates: it is optimal to serve *both* markets.

Key Result

$$p^U = \frac{3}{8}, \qquad q_A^U = \frac{5}{8}, \qquad q_B^U = \frac{1}{8}, \qquad \pi^U = \frac{9}{32}$$

Common Mistake to Avoid

Always verify whether the interior optimum lies within the feasible region $[0, \frac{1}{2}]$, *and* compare with the corner solution of dropping market B entirely. Failing to check the corner is a very common error.

Part (ii) — Third-degree price discrimination

Under discrimination, the firm sets independent prices p_A and p_B to maximise $\pi^D = \pi_A + \pi_B$.

Market A. $\pi_A = p_A(1 - p_A)$.

$$\text{FOC}_A: \quad 1 - 2p_A = 0 \implies p_A^D = \frac{1}{2}, \quad q_A^D = \frac{1}{2}, \quad \pi_A^D = \frac{1}{4}$$

Market B. $\pi_B = p_B\left(\frac{1}{2} - p_B\right)$.

$$\text{FOC}_B: \quad \frac{1}{2} - 2p_B = 0 \implies p_B^D = \frac{1}{4}, \quad q_B^D = \frac{1}{4}, \quad \pi_B^D = \frac{1}{16}$$

Key Result

$$p_A^D = \frac{1}{2}, \quad q_A^D = \frac{1}{2}, \qquad p_B^D = \frac{1}{4}, \quad q_B^D = \frac{1}{4}, \qquad \pi^D = \frac{1}{4} + \frac{1}{16} = \frac{5}{16} = \frac{10}{32}$$

Market A receives the higher price because its demand is less elastic.

Part (iii) — Welfare comparison

Consumer surplus on market i with linear demand $q_i = \bar{p}_i - p_i$ equals $\text{CS}_i = \frac{1}{2}(\bar{p}_i - p_i)^2$, where $\bar{p}_i$ is the choke price. Here $\bar{p}_A = 1$ and $\bar{p}_B = \frac{1}{2}$.

	Uniform (U)	**Discrimination** (D)	
CS_A	$\frac{1}{2}\left(\frac{5}{8}\right)^2 = \frac{25}{128}$	$\frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{16}{128}$	$\downarrow$
CS_B	$\frac{1}{2}\left(\frac{1}{8}\right)^2 = \frac{1}{128}$	$\frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{4}{128}$	$\uparrow$
CS (total)	$\frac{26}{128} = \frac{13}{64}$	$\frac{20}{128} = \frac{10}{64}$	$\downarrow$
Profit	$\frac{9}{32} = \frac{18}{64}$	$\frac{5}{16} = \frac{20}{64}$	$\uparrow$
Welfare	$\mathbf{\frac{31}{64}}$	$\mathbf{\frac{30}{64}}$	$\downarrow$

Economic Intuition

Relative to uniform pricing, third-degree price discrimination produces the following welfare effects:

- **Higher profit (Producer Surplus).** The monopolist gains two independent instruments (p_A, p_B) instead of one (p), allowing it to charge each segment its own monopoly price. Profit rises from 9/32 to 10/32.
- **Higher Consumer Surplus for the high-elasticity group (market B).** Because $\eta_B > \eta_A$, the discriminating price is *lower* for market B: $p_B^D = 1/4 < p^U = 3/8$. Consumers in market B gain, and CS_B rises from 1/128 to 4/128.
- **Lower Consumer Surplus for the low-elasticity group (market A).** The

discriminating price is *higher* for market A: $p_A^D = 1/2 > p^U = 3/8$. Consumers in market A lose, and CS_A falls from 25/128 to 16/128.

- **Lower Total Welfare.** The profit gain and the CS_B increase are not enough to offset the larger CS_A loss. Total surplus falls from 31/64 to 30/64.

Part (iv) — Does this result hold in general? The case $q_B = \frac{1}{3} - p_B$

General principle. The welfare-reducing result of part (iii) is *robust* whenever both markets are served under both pricing regimes. When a monopolist switches from uniform pricing to discrimination, if all markets continue to be served, the additional pricing instruments only allow more surplus extraction without any expansion of output — welfare unambiguously falls.

The result can be *overturned* only when discrimination brings a previously *excluded* market into play. In that case the output expansion generates new social surplus, and the overall welfare effect becomes ambiguous — it may rise or fall depending on which effect dominates.

New case: $q_B = \frac{1}{3} - p_B$. The choke price of market B is now $\frac{1}{3}$, making it even smaller.

Uniform pricing. For $p \le \frac{1}{3}$: aggregate demand $Q = \frac{4}{3} - 2p$, profit $\pi = p(\frac{4}{3} - 2p)$. FOC gives $p = \frac{1}{3}$, at which $q_B = 0$: market B is already at the margin. At $p = \frac{1}{2}$ (only market A): $\pi = \frac{1}{4}$. Comparing: $\pi(\frac{1}{3}) = \frac{2}{9} < \frac{1}{4}$. Therefore the firm sets $p^U = \frac{1}{2}$ and **excludes market B**.

Third-degree discrimination.

Market A: $p_A^D = \frac{1}{2}, \quad q_A^D = \frac{1}{2}$ Market B: $\text{FOC}_B: \frac{1}{3} - 2p_B = 0 \implies p_B^D = \frac{1}{6}, \quad q_B^D = \frac{1}{6}$

Under discrimination, market B is *served*; under uniform pricing, it is *not*. Total output therefore **increases** with discrimination.

	Uniform ($p^U = \frac{1}{2}$)	**Discrimination**
Total output	$\frac{1}{2}$	$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
CS_A	$\frac{1}{2}(\frac{1}{2})^2 = \frac{1}{8}$	$\frac{1}{2}(\frac{1}{2})^2 = \frac{1}{8}$
CS_B	0	$\frac{1}{2}\left(\frac{1}{6}\right)^2 = \frac{1}{72}$
Total CS	$\frac{1}{8} = \frac{9}{72}$	$\frac{9}{72} + \frac{1}{72} = \frac{10}{72}$
Profit	$\frac{1}{4} = \frac{18}{72}$	$\frac{18}{72} + \frac{2}{72} = \frac{20}{72}$
Welfare	$\frac{9}{72} + \frac{18}{72} = \frac{27}{72}$	$\frac{10}{72} + \frac{20}{72} = \frac{30}{72}$

Key Result

The general welfare implications of third-degree price discrimination relative to uniform pricing are:

- **Profit (Producer Surplus) is always higher.**
- **Consumer Surplus is higher** for the group with higher demand elasticity

(they face a lower discriminatory price) and **lower** for the group with lower elasticity (they face a higher price).

- **Total Welfare is lower** — *unless* total output increases. Output increases only when discrimination brings a previously excluded market into play (very rare). In the $q_B = \frac{1}{3} - p_B$ case, market B is excluded under uniform pricing but served under discrimination; total output expands and welfare **rises**.

Exercise C · PS II, Problem 1 — Cournot Oligopoly vs. Monopoly

Problem Statement

The market demand for a particular product is given by $p(Q) = 25 - Q$, where Q is the total quantity.

(i) Assume that the product is produced by a **single firm**. The total production cost faced by the firm is $C(Q) = 4Q$. Find the quantity that the firm will produce, the price that it will charge, and the profit that it will make in equilibrium.

(ii) Assume now that the same product is produced by **two firms**, firm 1 and firm 2. The total production cost faced by each firm i ($i = 1, 2$) is $C(q_i) = 4q_i$. The two firms choose their quantities *simultaneously and separately.* Find the quantity that each firm will produce, the price in the market, and each firm's profit in equilibrium.

(iii) Compare cases (i) and (ii) in terms of total quantity, market price, and consumer surplus.

Part (i) — Monopoly benchmark

The monopolist maximises:

$$\pi = (p - \text{MC})\,Q = (25 - Q - 4)\,Q = (21 - Q)\,Q$$

$$\text{FOC}: \quad 21 - 2Q = 0 \implies Q^m = \frac{21}{2} = 10.5$$

$$p^m = 25 - 10.5 = 14.5, \qquad \pi^m = (14.5 - 4) \times 10.5 = 110.25$$

$$\text{CS}^m = \frac{1}{2}(25 - 14.5) \times 10.5 = \frac{1}{2}(10.5)^2 = 55.125$$

Part (ii) — Cournot duopoly

Profit functions.

$$\pi_1 = (25 - q_1 - q_2 - 4)\,q_1 = (21 - q_1 - q_2)\,q_1$$
$$\pi_2 = (21 - q_1 - q_2)\,q_2$$

Best response functions.

$$\text{FOC}_1: \quad 21 - 2q_1 - q_2 = 0 \implies \text{BR}_1: \; q_1 = \frac{21 - q_2}{2} \tag{8}$$

$$\text{FOC}_2: \quad 21 - q_1 - 2q_2 = 0 \implies \text{BR}_2: \; q_2 = \frac{21 - q_1}{2} \tag{9}$$

The best response of each firm is a *decreasing* function of the rival's quantity: quantities are **strategic substitutes**.

Nash equilibrium. Substitute BR_2 into BR_1:

$$q_1 = \frac{21 - q_2}{2} = \frac{21 - \dfrac{21 - q_1}{2}}{2} = \frac{21 + q_1}{4} \implies 4q_1 = 21 + q_1 \implies q_1^* = 7$$

Substituting $q_1^* = 7$ back into BR_2: $q_2^* = \dfrac{21 - 7}{2} = 7$.

Key Result

$$q_1^* = q_2^* = 7, \qquad Q^C = 14, \qquad p^C = 11$$
$$\pi_1^* = \pi_2^* = (11 - 4) \times 7 = 49, \qquad \text{CS}^C = \tfrac{1}{2}(25 - 11) \times 14 = 98$$

Part (iii) — Comparison

	Monopoly	**Cournot Duopoly**	**Direction**
Total quantity Q	10.5	14	$\uparrow$
Price p	14.5	11	$\downarrow$
Consumer surplus	55.125	98	$\uparrow$
Total firm profit	110.25	98	$\downarrow$
Total welfare	165.375	196	$\uparrow$

Economic Intuition

Moving from monopoly to Cournot duopoly introduces competitive pressure: each firm's residual demand is constrained by the rival's output. The result is higher total output, a lower market price, greater consumer surplus, and higher total welfare — but lower industry profit. The Cournot equilibrium lies between the monopoly outcome and perfect competition on the market-power spectrum.

Exercise D · PS II, Problem 2 — Cournot Oligopoly with Asymmetric Marginal Costs

Problem Statement

Consider two firms that produce a homogeneous good and choose their quantities *simultaneously and separately.* The demand function is $P = a - q_1 - q_2$, where q_1 and q_2 are the quantities of firm 1 and firm 2 respectively. The cost functions are $C_1(q_1) = c_1 q_1$ and $C_2(q_2) = c_2 q_2$, where $c_1 < c_2$ and $c_2 < (a + c_1)/2$.

(i) Find the Nash equilibrium of the game. What are the market shares of the two firms?

(ii) Given your answer to (i), find the equilibrium profits, consumer surplus, and total surplus (welfare).

Part (i) — Nash equilibrium and market shares

Profit functions.

$$\pi_i = (a - q_1 - q_2 - c_i)\, q_i, \qquad i = 1, 2$$

FOCs and best response functions.

$$\text{FOC}_1: \quad a - 2q_1 - q_2 - c_1 = 0 \implies \text{BR}_1: \; q_1 = \frac{a - c_1 - q_2}{2} \tag{10}$$

$$\text{FOC}_2: \quad a - q_1 - 2q_2 - c_2 = 0 \implies \text{BR}_2: \; q_2 = \frac{a - c_2 - q_1}{2} \tag{11}$$

Solving the linear system. Rewriting as a 2×2 system:

$$2q_1 + q_2 = a - c_1 \tag{I}$$

$$q_1 + 2q_2 = a - c_2 \tag{II}$$

Multiply equation (I) by 2 and subtract equation (II) from the result (this eliminates q_2):

$$2(2q_1 + q_2) - (q_1 + 2q_2) = 2(a - c_1) - (a - c_2) \implies 3q_1 = a - 2c_1 + c_2 \implies \boxed{q_1^* = \frac{a - 2c_1 + c_2}{3}}$$

Multiply equation (II) by 2 and subtract equation (I) from the result (this eliminates q_1):

$$2(q_1 + 2q_2) - (2q_1 + q_2) = 2(a - c_2) - (a - c_1) \implies 3q_2 = a + c_1 - 2c_2 \implies \boxed{q_2^* = \frac{a - 2c_2 + c_1}{3}}$$

The condition $c_2 < (a + c_1)/2$ ensures $q_2^* > 0$.

Equilibrium price and market shares.

$$Q^* = \frac{2a - c_1 - c_2}{3}, \qquad P^* = \frac{a + c_1 + c_2}{3}$$

$$\alpha_1 = \frac{q_1^*}{Q^*} = \frac{a - 2c_1 + c_2}{2a - c_1 - c_2}, \qquad \alpha_2 = \frac{a - 2c_2 + c_1}{2a - c_1 - c_2}$$

Since $c_1 < c_2$: $q_1^* - q_2^* = c_2 - c_1 > 0$, so $\alpha_1 > \alpha_2$.

Key Result

$$q_i^* = \frac{a - 2c_i + c_j}{3}, \quad P^* = \frac{a + c_1 + c_2}{3}, \quad \alpha_1 > \alpha_2 \text{ because } c_1 < c_2$$

The low-cost firm captures a larger market share.

Common Mistake to Avoid

Do **not** assume symmetry here. Always solve the 2×2 system explicitly and verify the non-negativity condition $q_2^* > 0$.

Part (ii) — Profits, consumer surplus, and welfare

Profits. A useful Cournot identity: $P^* - c_i = (a - 2c_i + c_j)/3 = q_i^*$. Therefore:

$$\pi_i^* = (P^* - c_i)\, q_i^* = (q_i^*)^2$$

$$\pi_1^* = \left(\frac{a - 2c_1 + c_2}{3}\right)^2, \qquad \pi_2^* = \left(\frac{a - 2c_2 + c_1}{3}\right)^2$$

Consumer surplus and welfare.

$$\text{CS}^* = \frac{1}{2}(a - P^*)Q^* = \frac{(Q^*)^2}{2} = \frac{1}{2}\left(\frac{2a - c_1 - c_2}{3}\right)^2$$

$$W^* = \text{CS}^* + \pi_1^* + \pi_2^* = \frac{(Q^*)^2}{2} + (q_1^*)^2 + (q_2^*)^2$$

Economic Intuition

In Cournot competition with asymmetric costs, the market naturally allocates more output to the efficient firm — a form of productive efficiency. If firm 2's cost rises above $(a + c_1)/2$, it is driven out entirely and firm 1 acts as a monopolist.

Exercise E · PS II, Problem 6 — Bertrand Oligopoly with Asymmetric Costs

Problem Statement

Consider a market with duopoly in which the firms choose their prices *simultaneously and separately.* Market demand is $D(p) = 10 - p$. If a firm chooses a lower price than its competitor it supplies the whole market; if both firms choose the same price they share the market equally. The total costs of firm 1 and firm 2 are $C_1(q_1) = q_1$ and $C_2(q_2) = 4q_2$ respectively.
Find the Nash equilibrium prices as well as the equilibrium profits and output of each firm.

Preliminary: the Bertrand paradox with symmetric costs

In a symmetric Bertrand duopoly ($\text{MC}_1 = \text{MC}_2 = c$), both firms undercut each other until $p_1^* = p_2^* = c$, earning zero profit — the Bertrand paradox. Here $\text{MC}_1 = 1 \neq \text{MC}_2 = 4$: the symmetric benchmark no longer applies, and cost asymmetry breaks the paradox entirely, as we show below.

Step 1 — Is the asymmetry "small" or "large"?

If firm 1 faced no competition, it would set the monopoly price:

$$p_1^M = \frac{10+1}{2} = 5.5$$

Comparing with $\text{MC}_2 = 4$:

$$p_1^M = 5.5 > \text{MC}_2 = 4 \quad \Longrightarrow \quad \textbf{Small asymmetry}$$

In the small-asymmetry case firm 1 cannot charge its monopoly price without firm 2 entering profitably. (If $p_1^M \leq \text{MC}_2$, firm 1 would act as a pure monopolist — "large asymmetry" case.)

Step 2 — Nash equilibrium

Firm 1's strategy. Firm 1 undercuts firm 2 down to just below $\text{MC}_2 = 4$, where firm 2 can no longer match profitably.

Firm 2's strategy. At any $p_2 \geq \text{MC}_2 = 4$, firm 2 earns non-negative profit only if it attracts customers. Since firm 1 sets $p_1 < p_2$, firm 2 sells nothing.

In the limit ($\varepsilon \to 0$):

$$p_1^* = 4 - \varepsilon \approx \text{MC}_2, \qquad p_2^* = 4$$

$$q_1 \approx 10 - 4 = 6, \qquad q_2 = 0$$

$$\pi_1 \approx (4-1) \times 6 = 18, \qquad \pi_2 = 0$$

Key Result

$$p_1^* \approx 4, \quad q_1^* \approx 6, \quad \pi_1^* \approx 18 \qquad \text{and} \qquad p_2^* = 4, \quad q_2^* = 0, \quad \pi_2^* = 0$$

Firm 1 acts as a **de facto monopolist** at a price equal to MC_2.

Common Mistake to Avoid

A common error is to set $p_1^* = \text{MC}_1 = 1$. That is correct only in the *symmetric* Bertrand model. Here, firm 1 exploits its cost edge by pricing just below MC_2, not at its own marginal cost. Always check whether the asymmetry is small or large first.

Economic Intuition

Cost asymmetry resolves the Bertrand paradox: even with homogeneous goods and simultaneous price-setting, a cost advantage generates positive equilibrium profit. Firm 1 uses its lower cost as a competitive weapon, earning a margin of $\text{MC}_2 - \text{MC}_1 = 3$ per unit on roughly 6 units.

Exercise F · PS II, Problem 7 — Bertrand Oligopoly with Differentiated Products

Problem Statement

Suppose a market consists of two firms, firm 1 and firm 2, who are Bertrand competitors. Firm 1 makes and sells good 1 and firm 2 makes and sells good 2. The demand functions are:

$$q_1 = 25 - 5p_1 + 2p_2 \qquad \text{and} \qquad q_2 = 25 - 5p_2 + 2p_1$$

The total cost functions are $C(q_1) = 2 + q_1$ and $C(q_2) = 2 + q_2$.

(i) Find the best response function of each firm. How does the price firm i sets change with increases in the price of its competitor's good? What is the Nash equilibrium?

(ii) What is the Lerner Index for good 1 and for good 2? What is the interpretation? Do the firms have market power?

(iii) Explain why the Bertrand Paradox of zero market power does not apply in this case.

Part (i) — Best response functions and Nash equilibrium

Firm 1's profit. The total cost is $C(q_1) = 2 + q_1$, so $\text{MC} = 1$ and there is a fixed cost of 2 (which does not affect the first-order conditions but does affect the profit level):

$$\pi_1 = (p_1 - 1)(25 - 5p_1 + 2p_2) - 2$$

FOC for firm 1.

$$\frac{\partial \pi_1}{\partial p_1} = (25 - 5p_1 + 2p_2) + (p_1 - 1)(-5) = 0 \implies 30 + 2p_2 - 10p_1 = 0$$

$$\text{BR}_1: \quad p_1 = 3 + \frac{p_2}{5} \tag{12}$$

Firm 2's profit. The cost structure is identical for firm 2: $C(q_2) = 2 + q_2$, $\text{MC} = 1$, fixed cost 2.

$$\pi_2 = (p_2 - 1)(25 - 5p_2 + 2p_1) - 2$$

FOC for firm 2.

$$\frac{\partial \pi_2}{\partial p_2} = (25 - 5p_2 + 2p_1) + (p_2 - 1)(-5) = 0 \implies 30 + 2p_1 - 10p_2 = 0$$

$$\text{BR}_2: \quad p_2 = 3 + \frac{p_1}{5} \tag{13}$$

Interpretation. The slope $+\frac{1}{5} > 0$ means prices are **strategic complements**: if firm 2 raises its price, firm 1 optimally raises its own price too. This is the opposite of Cournot quantities (strategic substitutes).

Nash equilibrium. Substituting BR_2 into BR_1:

$$p_1^* = 3 + \tfrac{1}{5}\Big(3 + \tfrac{p_1^*}{5}\Big) \implies p_1^* \cdot \tfrac{24}{25} = \tfrac{18}{5} \implies p_1^* = \frac{15}{4} = 3.75$$

Substituting $p_1^* = \frac{15}{4}$ back into BR_2:

$$p_2^* = 3 + \frac{p_1^*}{5} = 3 + \frac{15/4}{5} = 3 + \frac{3}{4} = \frac{15}{4}$$

Equilibrium quantities and profits:

$$q_i^* = 25 - 5 \cdot \tfrac{15}{4} + 2 \cdot \tfrac{15}{4} = \frac{55}{4} = 13.75$$

$$\pi_i^* = \left(\frac{15}{4} - 1\right) \cdot \frac{55}{4} - 2 = \frac{11}{4} \cdot \frac{55}{4} - 2 = \frac{605}{16} - \frac{32}{16} = \frac{573}{16} \approx 35.81$$

Key Result

$$p_1^* = p_2^* = \frac{15}{4} = 3.75, \qquad q_1^* = q_2^* = \frac{55}{4} = 13.75, \qquad \pi_1^* = \pi_2^* = \frac{573}{16} \approx 35.81$$

Part (ii) — Lerner index

$$L = \frac{p^* - \text{MC}}{p^*} = \frac{15/4 - 1}{15/4} = \frac{11/4}{15/4} = \frac{11}{15} \approx 73\%$$

Key Result

$L_1 = L_2 = 11/15 \approx 73\%$. Both firms have substantial market power.

To verify the Lerner condition $L = 1/\eta$ (with $\eta = -\partial q_1/\partial p_1 \cdot p_1/q_1$):

$$\eta_1 = -(-5) \cdot \frac{15/4}{55/4} = 5 \cdot \frac{15}{55} = \frac{15}{11} \implies \frac{1}{\eta_1} = \frac{11}{15} = L_1 \quad \checkmark$$

Part (iii) — Why the Bertrand paradox does not apply

The Bertrand paradox requires that consumers always buy from the cheapest supplier, so any firm pricing above MC instantly loses the entire market. Here, goods 1 and 2 are **imperfect substitutes**: the $+2p_j$ term means some consumers strictly prefer good i regardless of small price differences. Consequently:

1. **Undercutting does not steal the whole market.** Cutting p_1 by ε attracts *some* but not *all* of firm 2's customers. The incentive to undercut to MC vanishes.
2. **Each firm retains market power.** With captive consumers, a firm can price above MC and earn positive profit in equilibrium.
3. **Prices are strategic complements.** Firms raise prices in response to rivals' price increases, reinforcing positive margins rather than racing to the bottom.

Economic Intuition

Product differentiation restores market power even under price competition. As goods become closer substitutes (the cross-price coefficient rises relative to the own-price coefficient), the Lerner index falls and the equilibrium approaches the Bertrand paradox outcome of $p = \text{MC}$.