ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ
ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Department of International and European Economic Studies

INDUSTRIAL ECONOMICS

Extra Practice Exercises Solutions

Chrysovalantou Milliou (cmilliou@aueb.gr)

Exercise 1

Consider a market with three firms producing a homogeneous good and facing inverse demand function $p(Q) = 1-Q$, where $Q = q_1+q_2+q_3$ denotes aggregate quantity. All firms constant marginal cost of production given by c. The three firms choose their quantities simultaneously and separately.

1. Find each firm's best response function and interpret it.
2. Find the equilibrium quantities, price and profits.

Solution

(1) To find the best response functions we need to solve the profit maximization problem of each firm $i = 1, 2, 3$, taking the quantities of other firms $j \neq i$ as given.

$$\max_{q_i \geq 0} \Pi_i(q_i) = (1 - q_i - \sum_{j \neq i} q_j)q_i - cq_i$$

For brevity, represent the total quantity produced by firms other than i by q_{-i}, i.e.

$$q_{-i} = \sum_{j \neq i} q_j$$

So the problem becomes

$$\max_{q_i \geq 0} \Pi_i(q_i) = (1 - q_i - q_{-i})q_i - cq_i$$

The first order condition is

$$\frac{\partial \Pi_i(q_i)}{\partial q_i} = 0 \Rightarrow 1 - 2q_i - q_{-i} - c = 0 \Rightarrow \boxed{q_i(q_{-i}) = \frac{1-c}{2} - \frac{1}{2}q_{-i}}$$

This is the best response function of firm i. It is a product of firm i's maximization problem, and thus it shows us the optimal behavior of firm i given the behavior of other firms, i.e. what quantity i should produce to maximize its profits, given the quantities produced by other firms.

Notice that taking the derivative of the best response function with respect to q_{-i} we get that there is a negative relationship between q_i and q_{-i}. This property of quantities as strategic variables in the Cournot model, i.e. that a firm finds it optimal to do the opposite of its opponent (e.g. increase its quantity as a response to a decrease in quantity) is called *strategic substitutability*. (Contrast this with the classic Bertrand models, in which prices are strategic complements, instead of substitutes.) □

(2) The best response functions that we found in (1) form a linear system of 3 equations in 3 unknowns, which has a unique solution, thus there is a unique Nash equilibrium. Here is the system explicitly,

$$\begin{aligned} q_1 &= \frac{1-c}{2} - \frac{1}{2}(q_2+q_3) \\ q_2 &= \frac{1-c}{2} - \frac{1}{2}(q_1+q_3) \\ q_3 &= \frac{1-c}{2} - \frac{1}{2}(q_1+q_2) \end{aligned}$$

Note that this system is symmetric, thus its unknowns must be equal. Therefore in equilibrium, all firms produce the same amount, $q_1^* = q_2^* = q_3^* = q^*$. This allows us to solve the system very quickly (without using substitution or matrix methods).

$$q^* = \frac{1-c}{2} - \frac{1}{2}(2q^*) \Rightarrow 2q^* = 1-c-2q^* \Rightarrow$$

$$\Rightarrow 4q^* = 1-c \Rightarrow \boxed{q^* = \frac{1-c}{4}}$$

Now, we plug this into the demand function to find the price in equilibrium,

$$p^* = 1 - \frac{3(1-c)}{4} \Rightarrow \boxed{p^* = \frac{1+3c}{4}}$$

Finally, we can find the profits of each firm in equilibrium,

$$\Pi^* = \frac{1+3c}{4}\frac{1-c}{4} - c\frac{1-c}{4} \Rightarrow \boxed{\Pi^* = \frac{(1-c)^2}{16}} \quad \blacksquare$$

Exercise 2

In an oligopolistic market, there are initially four firms (1,2,3 and 4). The total cost of production of each firm i, with $i = 1,2,3,4$ is equal to $C(q_i) = 3q_i$, where q_i is the quantity of firm i. The firms choose their quantities simultaneously and independently. The total market demand for their product is $p(Q) = 18 - Q$, where Q is the total quantity of the product.

1. Find the quantity and profits of each firm in equilibrium.
2. Suppose now that firms 1,2, and 3 are considering merging with each other. The total cost of production of the merged firm will be $C(q_m) = 3q_m$, where q_m is the quantity of output of the merged firm. Do the three firms have an incentive to merge or not?
3. Suppose now that if firms 1,2, and 3 merge, the total cost of production of the merged firm is $C(q_m) = 2q_m$, where q_m is the quantity of the merged firm. Do the three firms have an incentive to merge or not?

Solution

(1) Like in exercise (1), we need to solve the profit maximization problem of each firm, in order to find their best response functions

$$\max_{q_i \geq 0} \Pi_i(q_i) = (18 - q_i - q_{-i})q_i - 3q_i$$

After a little algebra, the problem becomes

$$\max_{q_i \geq 0} \Pi_i(q_i) = (15 - q_i - q_{-i})q_i$$

The first order condition is

$$\frac{\partial \Pi_i(q_i)}{\partial q_i} = 0 \Rightarrow 15 - 2q_i - q_{-i} = 0 \Rightarrow \boxed{q_i(q_{-i}) = \frac{15 - q_{-i}}{2}}$$

This is firm i's best response function. We have a linear system of 4 equations in 4 unknowns. As in exercise (1) the system is again symmetric, so we can just set $q_1^* = q_2^* = q_3^* = q_4^* = q^*$ to find $\boxed{q^* = 3.}$ Then by plugging this into the demand function we get the equilibrium price $\boxed{p^* = 6,}$ and finally we can find the profits of each firm in equilibrium $\boxed{\Pi^* = 9.}$ □

(2) After the merger we have two firms in the market, the merged firm (m) and firm 4. The merged firm solves

$$\max_{q_m} \Pi_m(q_m) = (18 - q_m - q_4)q_m - 3q_m = 15q_m - q_m^2 - q_m q_4$$

The first order condition is

$$\frac{\partial \Pi_m(q_m)}{\partial q_m} = 0 \Rightarrow 15 - 2q_m - q_4 = 0 \Rightarrow \boxed{q_m(q_4) = \frac{15 - q_4}{2}}$$

Firm 4 solves

$$\max_{q_4} \Pi_4(q_4) = (18 - q_4 - q_m)q_4 - 3q_4 = 15q_4 - q_4^2 - q_4 q_m$$

The first order condition is

$$\frac{\partial \Pi_4(q_4)}{\partial q_4} = 0 \Rightarrow 15 - 2q_4 - q_m = 0 \Rightarrow \boxed{q_4(q_m) = \frac{15 - q_m}{2}}$$

As we can see (both from the profit function and the best responses), the firms are symmetric when it comes to their quantity setting behavior. Their only difference is their ownership structure, i.e. that the profits of the merged firm have to be shared between firms 1, 2, and 3, while firm 4 is entitled to all of its profits. Ownership structure does not play a role in how the firm decides its quantity. What matters is joint profit maximization. (Maximizing the size of the pie in this model has nothing to do with how the pie is shared.)

Due to symmetry, we can easily solve the system to find

$$\boxed{q_m^* = q_4 = 5^* \Rightarrow p^* = 8 \Rightarrow \Pi_m^* = \Pi_4^* = 25}$$

The merger will take place if the profits of the merged firm are higher than the profits firm 1, 2, and 3 independently make before the merger. Here, the merger does not take place, because before the merger the profit of an independent firm is 9. Hence, firms 1, 2, and 3 collectively realize profit equal to 27 before the merger, which is higher than 25, their profit if they merge.

This result is known in the literature as the *merger paradox*. The merger we examined is a merger between firms that have a very high share in the market, 75%, so it is not desirable that the model predicts this merger is unprofitable, because in reality we observe mergers between firms with much less share in the market. The reason for this paradox is attributed to the only effect of this merger being the reduction of the number of firms, which leads to higher profits for each firm due to higher concentration. However, as those increased profits have to be shared between three partners, they are not enough to incentivize the merger. Therefore, we need to find other ways in which the merger gives advantages to the merged firm. (This is the point of the following question.) □

(3) Now the merged firm has reduced costs, because the merger is assumed to be efficiency enhancing. The new maximization problem of the merged firm is

$$\max_{q_m} \Pi_m(q_m) = (18 - q_m - q_4)q_m - 2q_m = 16q_m - q_m^2 - q_m q_4$$

The first order condition is

$$\frac{\partial \Pi_m(q_m)}{\partial q_m} = 0 \Rightarrow 16 - 2q_m - q_4 = 0 \Rightarrow \boxed{q_m(q_4) = \frac{16 - q_4}{2}}$$

Firm 4's problem is the same as in (2),

$$\max_{q_4} \Pi_4(q_4) = (18 - q_4 - q_m)q_4 - 3q_4 = 15q_4 - q_4^2 - q_4 q_m$$

The first order condition is

$$\boxed{q_4(q_m) = \frac{15 - q_m}{2}}$$

The best response functions form a linear system of 2 equations in 2 unknowns, which is unfortunately not symmetric. We solve by substitution (plug one of the best response functions into the other) to find

$$q_m^* = 5.67, q_4^* = 4.66, p^* = 7.67, \Pi_m^* = 32.35$$

Notice that $32.35 \geq 27$ hence the merger will take place. The reduction in costs is large enough to incentivize the merger. ■

Exercise 3

Consider an industry with 3 firms competing à la Cournot. Every firm $i = 1, 2, 3$ faces an inverse demand function:

$$p_i(q_i, q_{-i}) = 1 - q_i - \gamma q_{-i}$$

where

$$q_{-i} = \sum_{j \neq i} q_j$$

is the total quantity produced by every firm except for i, and $\gamma \in [0, 1]$ represent the degree of product differentiation. When $\gamma = 0$, products are completely differentiated, but when $\gamma = 1$, products are homogeneous. For simplicity, suppose that firms have constant marginal costs of production, $c = 0$.

1. *Cournot competition.* Assume that every firm i independently and simultaneously chooses its output level q_i. Find the equilibrium output, price, and profits for each firm.
2. *Full merger.* Assume that all three firms merge to choose an output profile (q_1, q_2, q_3) that maximizes their joint profits. Find the equilibrium output, price, and profits in this setting.
3. *Partial merger.* Assume that firm 1 and firm 2 merge to choose an output profile (q_1, q_2) that maximizes their joint profits. Find the equilibrium output.
4. Evaluate the equilibrium output in (3) when $\gamma = 0$ and when $\gamma = 1$. Interpret.
5. Find the equilibrium price and profits of merged and unmerged firms in (3).
6. Evaluate the equilibrium profits in (5) when $\gamma = 0$ and when $\gamma = 1$. Interpret.
7. Under what condition on γ can the partial merger in (3) be sustained?

Solution

(1) *Important Note: This market may be a market of differentiated goods, which is different than the homogeneous good market of exercises (1) and (2), however our solution approach remains similar. It is still a Cournot model, so we write profit as a function of quantity. It is still a simultaneous move game, so our goal is to find the Nash equilibrium. The profit functions still have desirable properties (continuity, differentiability etc.), so again we find the Nash equilibrium by finding the best response functions and solving the system of equations they form.*

Firm $i = 1, 2, 3$ solves

$$\max_{q_i \geq 0} \Pi_i(q_i) = (1 - q_i - \gamma q_{-i})q_i$$

The first order condition is

$$\frac{\partial \Pi_i(q_i)}{\partial q_i} = 0 \Rightarrow 1 - 2q_i - \gamma q_{-i} = 0 \Rightarrow \boxed{q_i(q_{-i}) = \frac{1 - \gamma q_{-i}}{2}}$$

This is the best response function of firm i. The best response functions form a symmetric linear system of 3 equations in 3 unknowns. Due to symmetry, we can quickly find the equilibrium by setting $q_1^* = q_2^* = q_3^* = q^*$.

$$q^* = \frac{1 - 2\gamma q^*}{2} \Rightarrow \boxed{q^* = \frac{1}{2(1+\gamma)}}$$

Plugging this to the demand function we get the equilibrium price

$$p^* = 1 - \frac{1}{2(1+\gamma)} - \frac{2\gamma}{2(1+\gamma)} \Rightarrow \boxed{p^* = \frac{1}{2(1+\gamma)}}$$

Finally the profits of each firm in equilibrium are

$$\boxed{\Pi^* = \frac{1}{4(1+\gamma)^2}} \square$$

(2) If all firms merge we have a multi-product monopoly. The merged firm solves

$$\max_{\{q_1, q_2, q_3\}} \Pi_m(q_1, q_2, q_3) = \sum_{i=1}^{3} [(1 - q_i - \gamma q_{-i})q_i]$$

This problem has 3 first order conditions, one for each q_i. To take the derivatives it might help to expand the sum as follows

$$\Pi_m(q_1, q_2, q_3) = (1 - q_1 - q_2 - q_3)q_1 + (1 - q_2 - q_1 - q_3)q_2 + (1 - q_3 - q_1 - q_2)q_3$$

It should be clear that the first order condition for q_i is

$$\frac{\partial \Pi_m(q_1, q_2, q_3)}{\partial q_i} = 0 \Rightarrow 1 - 2q_i - 2\gamma q_{-i} = 0 \Rightarrow \boxed{q_i = \frac{1}{2} - \gamma q_{-i}}$$

Notice that this condition is similar to the best response function we found in (1), with the following difference. The coefficient of q_{-i} is $-\gamma$ instead of $-\gamma/2$. The reason for this is that the merged firm realizes that an increase in, say, q_1, does not only affect the profits made by selling product 1, but it also affects the profits made by selling the other two products. The independent firms in (1) do not internalize this externality, because they only care about their own profits.

The first order conditions form a symmetric linear system of 3 equations in three unknowns. We can set $q_1^* = q_2^* = q_3^* = q^*$ to find

$$\boxed{q^* = \frac{1}{2(1+2\gamma)},\ p^* = \frac{1}{2},\ \Pi^* = \frac{1}{4(1+2\gamma)}} \square$$

(3) If firms 1 and 2 merge, and 3 stays independent, then we have two firms in the market, the merged firm and firm 3. The merged firm is a multi-product firm, while firm 3 is a single-product firm. The merged firm solves

$$\max_{\{q_1, q_2\}} \Pi_m = \sum_{i=1}^{2} [(1 - q_i - \gamma q_{-i})q_i] = \sum_{i=1}^{2} [(1 - q_i - \gamma q_j - \gamma q_3)q_i]$$

where $i, j = 1, 2$ and $j \neq i$. Here we have two first order conditions, one for q_1 and one for q_2.

$$\frac{\partial \Pi_m(q_1, q_2)}{\partial q_i} = 0 \Rightarrow \boxed{q_i = \frac{1 - 2\gamma q_j - \gamma q_3}{2}}$$

The two first order conditions form a symmetric linear system of 2 equations with 2 unknowns. We set $q_1^* = q_2^* = q_m^*$ to find

$$\boxed{q_m^* = \frac{1 - \gamma q_3^*}{2(1+\gamma)}}$$

This is the best response function of the merged firm.

Firm 3 solves

$$\max_{q_3} \Pi_3(q_3) = (1 - q_3 - \gamma q_1 - \gamma q_2) q_3$$

The first order condition is

$$\frac{\partial \Pi_3(q_3)}{\partial (q_3)} = 0 \Rightarrow 1 - 2q_3 - \gamma q_1 - \gamma q_2 = 0 \Rightarrow \boxed{q_3 = \frac{1 - \gamma q_1 - \gamma q_2}{2}}$$

This is the best response function of firm 3. In equilibrium, this best response function becomes

$$\boxed{q_3^* = \frac{1 - 2\gamma q_m^*}{2}}$$

since $q_1^* = q_2^* = q_m^*$.

The two best response functions form a linear system of 2 equations in 2 unknowns. Unfortunately it is not symmetric so we solve by substitution to find

$$\boxed{q_m^* = \frac{2 - \gamma}{2(2 + 2\gamma - \gamma^2)}}$$

$$\boxed{q_3^* = \frac{2}{2(2 + 2\gamma - \gamma^2)}} \quad \square$$

(4) If $\gamma = 0$ the goods are completely differentiated, i.e. there is no degree of substitution between them for the consumers. We find

$$q_m^* = q_3^* = \frac{1}{2}$$

which is the typical monopoly solution (demands are independent of each other).

If $\gamma = 1$ the goods are homogeneous, i.e. they are perfect substitutes. We find

$$q_m^* = \frac{1}{6}, \; q_3^* = \frac{1}{3}$$

Recall that $q_m^* = q_1^* = q_2^*$ hence the total quantity produced by the merged firm $(q_1^* + q_2^*)$ is $1/3$, exactly the same as firm 3 (as we expect from exercises (1) and (2) of the problem set). $\square$

(5) We substitute the quantities we found in (3) into the demand function of each firm. First for the merged firm

$$p_m^* = 1 - q_m^* - \gamma q_m^* - \gamma q_3^* - = 1 - \frac{2 - \gamma}{2(2 + 2\gamma - \gamma^2)} - \frac{\gamma(2-\gamma)}{2(2 + 2\gamma - \gamma^2)} - \frac{2\gamma}{2(2 + 2\gamma - \gamma^2)} \Rightarrow$$

$$\Rightarrow \boxed{p_m^* = \frac{2 + \gamma - \gamma^2}{2(2 + 2\gamma - \gamma^2)}}$$

Then for firm 3,

$$p_3^* = 1 - q_3^* - \gamma q_m^* - \gamma q_m^* = 1 - q_3^* - 2\gamma q_m^* \Rightarrow$$

$$\Rightarrow p_3^* = 1 - \frac{2}{2(2+2\gamma-\gamma^2)} - \frac{2\gamma(2-\gamma)}{2(2+2\gamma-\gamma^2)} \Rightarrow$$

$$\Rightarrow \boxed{p_3^* = \frac{2+\gamma-\gamma^*2}{2(2+2\gamma-\gamma^2)}}$$

The profits of the merged firm are

$$\Pi_m^* = p_1^* q_1^* + p_2^* q_2^* = 2p_m^* q_m^* = 2\frac{2+\gamma-\gamma^2}{2(2+2\gamma-\gamma^2)}\frac{2-\gamma}{2(2+2\gamma-\gamma^2)} \Rightarrow$$

$$\Rightarrow \boxed{\Pi_m^* = \frac{(2+\gamma-\gamma^2)(2-\gamma)}{2(2+2\gamma-\gamma^2)^2}} \square$$

(6) If $\gamma = 0$ the goods are completely differentiated. We find

$$\Pi_m^* = \frac{1}{2}, \ \Pi_3^* = \frac{1}{4}$$

which is the typical monopoly solution (demands are independent of each other if $\gamma = 0$). Note that the merged firm operates two separate monopolies, and this is why it has double the profits of firm 3.

If $\gamma = 1$ the goods are homogeneous. We find

$$\Pi_m^* = \Pi_3^* = \frac{1}{9}$$

which is expected, since being a multi-product firm in a homogeneous good environment is irrelevant - the merged firm and firm 3 are symmetric.

(7) For the merger to take place, the profits made by the merged firm in (3) must be higher than the sum of profits of firm 1 and 2 in (1).

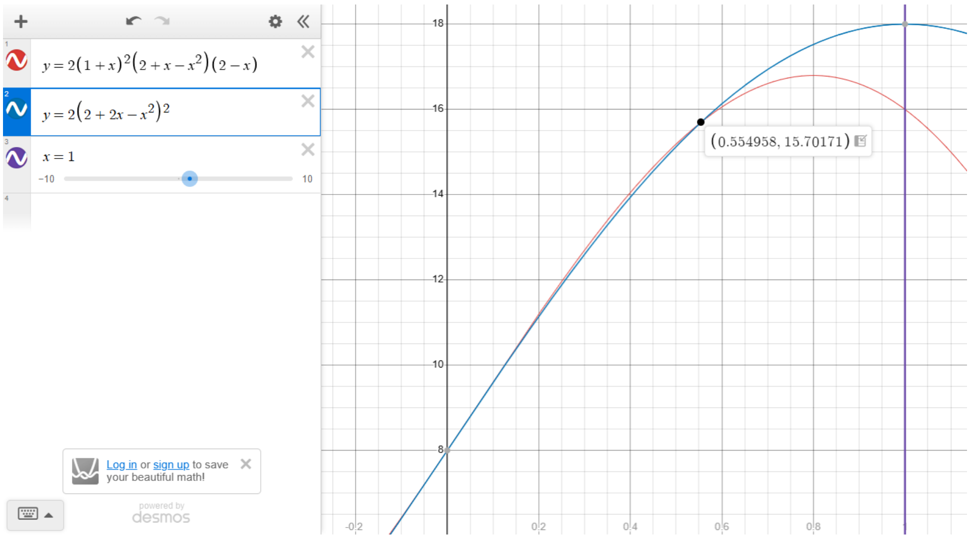
$$\frac{(2+\gamma-\gamma^2)(2-\gamma)}{2(2+2\gamma-\gamma^2)^2} \geq \frac{1}{4(1+\gamma)^2} + \frac{1}{4(1+\gamma)^2} \Rightarrow$$

$$\Rightarrow \frac{(2+\gamma-\gamma^2)(2-\gamma)}{2(2+2\gamma-\gamma^2)^2} \geq \frac{1}{2(1+\gamma)^2} \Rightarrow$$

$$\Rightarrow (1+\gamma)^2(2+\gamma-\gamma^2)(2-\gamma) \geq (2+2\gamma-\gamma^2)^2$$

It can be proven that there is a critical point γ_c such that if $\gamma < \gamma_c$ then the merger is profitable and takes place, while if $\gamma > \gamma_c$ then the merger is unprofitable. This is interpreted as follows. In this model, contrary to the models of the first two exercises of this problem set, since we have differentiated goods, the merged firm becomes a multi-product firm. This makes it have an advantage over firm 3. The higher the degree of differentiation (i.e. the lower γ is), the higher the multi-product advantage of the merged firm, since the consumers it serves have a lower degree of substitution between the two goods it produces.

The red line in the following graph is the left hand side of the inequality, while the blue line is the right hand side. We can see that this critical point is at $\gamma = 0.554958$



■

Exercise 4

Consider two firms (1,2) competing à la Bertrand (choosing their prices). Assume that they both face constant marginal cost 0.

1. Suppose that this is a homogeneous product market and that firms choose their prices simultaneously. The demand function is:

$$q_i(p_i, p_j) = \begin{cases} 10 - p_i, & \text{if } p_i < p_j \\ \dfrac{10 - p_i}{2}, & \text{if } p_i = p_j \\ 0, & \text{if } p_i > p_j \end{cases}$$

   where $i \neq j = 1, 2$. What are the equilibrium price levels? Find the profits of each firm in equilibrium.

2. Suppose now that this is a differentiated product market. Firms still choose their prices simultaneously. The demand function is now:

$$q_i(p_i, p_j) = 10 - p_i + gp_j,$$

   where $g > 0$ is a product differentiation parameter. (Note that if $g = 0$, the products are completely differentiated, and when $g \to \infty$, the products are completely homogeneous.) What are the equilibrium price levels? Find the profits of each firm in equilibrium.

3. In the differentiated product setting described in part (2), assume now that firm 1 chooses its price before firm 2, and that firm 2 observes firm 1's price before choosing its own. What are the equilibrium price levels? Find the profits of each firm in equilibrium. Compare with part (2). Does a firm prefer to choose its price first or second?

Solution

(1) *Note: The homogeneous Bertrand model is still a simultaneous-move game, so we need to find the Nash equilibrium, but it has the following difference from the Cournot models we have seen in the previous exercises of this problem set: The profit function does not have desirable properties as it is discontinuous. For this reason, it is inconvenient to solve the problem using optimization methods to get the best response functions, and we have to go with the definition of the Nash Equilibrium (a strategy profile - here a price for each firm - such that no player has incentive to deviate, given the strategies of the other players). Intuitively we know that the Nash* *equilibrium of this game is $p_1^* = p_2^* = 0$. We will prove here that this is in fact a Nash equilibrium, and that it is unique.*

$(p_1, p_2) = (0, 0)$ is a Nash equilibrium of this game because no player has incentive to deviate. In this equilibrium, the two firms share the market - each firm sells 5 units. The profits in this equilibrium are $(\Pi_1, \Pi_2) = (0, 0)$. If a firm, say firm 1, increases its price, then it sells no units, thus its profits remain 0, so there is no incentive to deviate.

Now we will show that (p_1, p_2) with $p_1 \neq p_2$ is not a Nash equilibrium. Suppose, without loss of generality (just reverse the subscripts and apply the same arguments) that $p_1 < p_2$. This means that firm 1 sells $10 - p_1$ units at price p_1 and makes profit $p_1(10 - p_1)$. Firm 2 sells nothing and makes no profit. In this strategy profile, firm 1 has an incentive to deviate, by increasing its price to $p_2 - \epsilon$ where ϵ is an infinitesimally small positive number.

Finally we will show that (p_1, p_2) with $p_1 = p_2 = p > 0$ is not a Nash equilibrium. Here, each firm sells $(10 - p)/2$ units at price p and makes profit $p(10 - p)/2$. Both firms have incentive to deviate by marginally decreasing their price to $p - \epsilon$, taking 100% share of the market, selling $10 - p - \epsilon$ units, which leads to profit $(p - \epsilon)(10 - p - \epsilon)$ which are greater that $p(10 - p)/2$. □

(2) *Note: Here we have a differentiated goods Bertrand model. This has profit functions with desirable properties, so we find the Nash Equilibrium like in the Cournot model, by finding the best response functions and solving the system they form. The difference with Cournot is that the strategic variables are prices, hence we write the profits in terms of price.*

Firm $i = 1, 2$ solves

$$\max_{p_i} \Pi_i(p_i) = (10 - p_i - gp_j)p_i$$

with $j \neq i$. Taking first order conditions we find

$$\frac{\partial \Pi_i}{\partial p_i} = 0 \Rightarrow 10 - 2p_i + gp_j = 0 \Rightarrow \boxed{p_i(p_j) = \frac{10 + gp_j}{2}}$$

This is the best response function of firm i. Note that p_i has a positive relationship with p_j, i.e. when firm j increases its price, firm i then wants to increase its price. This property of prices in the Bertrand model is called *strategic complementarity.* (Contrast with quantities in Cournot model, which are strategic substitutes.)

The two best response functions form a symmetric linear system of 2 equations in 2 unknowns. Due to symmetry we can easily solve by setting $p_1^* = p_2^* = p^*$.

$$p^* = \frac{10 + gp^*}{2} \Rightarrow p^*(1 - \frac{g}{2}) = 5 \Rightarrow p^* \frac{2 - g}{2} = 5 \Rightarrow \boxed{p^* = \frac{10}{2 - g}}$$

Plugging this into the demand function we get

$$q^* = 10 - \frac{10}{2 - g} + g\frac{10}{2 - g} = \cdots = \frac{10}{2 - g}$$

which is the quantity each firm produces in equilibrium. Finally the profits of each firm are

$$\Pi^* = p^* q^* = \left(\frac{10}{2 - g}\right)^2 \quad \square$$

(3) *Note: Here we have a differentiated goods price-setting game like before, but it is not simultaneous. One firm moves first and the other second (like a Stackelberg model, but with prices as strategic variables instead of quantities). We need to find the subgame-perfect Nash equilibrium (SPNE) through backwards induction. This is done by solving the problem of the firm that moves second (follower), finding its best response function, and plugging this best response function to the problem of the firm that moves first (leader)* ***before we take first order conditions****.*

The follower solves

$$\max_{p_2} \Pi_2(p_2) = (10 - p_2 + gp_1)p_2$$

which leads to the best response function

$$p_2(p_1) = \frac{10 - gp_1}{2}$$

(I have skipped a few steps since this is the same problem as in (2)).

The leader solves

$$\max_{p_1} \Pi_1(p_1) = (10 - p_1 + g\frac{10 - gp_1}{2})p_1 = \frac{20 - 2p_1 + 10g + g^2 p_1}{2} p_1$$

Taking first order conditions,

$$\frac{\partial \Pi_1(p_1)}{\partial p_1} = 0 \Rightarrow \frac{1}{2}(20 - 4p_1 + 10g + 2g^2 p_1) = 0 \Rightarrow$$

$$\Rightarrow p_1(2 - 2g^2) = 20 + 10g \Rightarrow 2p_1(2 - g^2) = 10(2 + g) \Rightarrow \boxed{p_1^* = \frac{5(2+g)}{2 - g^2}}$$

Then we plug this into the best response function of the follower to find

$$p_2^* = \frac{5(4 + 2g - g^2)}{2(2 - g^2)}$$

Plugging into the demand functions we get

$$q_1^* = \frac{5(2+g)}{2}, \ q_2^* = \frac{5(4 + 2g - g^2)}{2(2 - g^2)}$$

Finally we find equilibrium profits

$$\Pi_1^* = \frac{25(2+g)^2}{2(2 - g^2)}, \ \Pi_2^* = \frac{25(4 + (2-g)g)^2}{4(2 - g^2)^2}$$

Notice that

$$\Pi_1^* - \Pi_2^* = \frac{-25g^3(4 + 3g)}{4(2 - g^2)^2}$$

which is less than zero. So the leader makes less than the follower, and it would prefer to be a follower instead. This is called second mover's advantage.

Also contrast with the Stackelberg model, where we have dynamic price setting and strategic substitutability, which leads to a first-mover's advantage. ■