Revision Matrices

Let A be a nxk Matrix with elements a_{ij} for i=1,2,…,n and j=1,2,…,k, and B a matrix of the same order, i.e. nxk, and elements b_{ij} for i=1,2,…,n and j=1,2,…,k.
If n=k, i.e. the number of rows is equal to the number of columns, the matrix is called **square**.

We define the **sum** of the two matrices as a nxk matrix C with elements c_{ij} for i=1,2,…,n and j=1,2,…,k such that

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for } i = 1,2,...,n \text{ and } j = 1,2,..k.$$

Notice that the orders of A and B must be the same.

Let D be a kxq matrix with elements d_{ij} for i=1,2,…,k and j=1,2,…,q. Then the product of A and D is a matrix, say AD=F, of order nxq with elements f_{ij} for i=1,2,…,n and j=1,2,…,q such that

$$f_{ij} = \sum_{m=1}^{k} a_{im} d_{mj} + \quad \text{for } i = 1,2,...,n \text{ and } j = 1,2,..q.$$

Notice that the f_{ij} is the inner product of the i^{th} row of A and the j^{th} column of D. Further, the product is define only for matrices that the left term of the product has the same number of columns as the number of rows of the right term. It follows that DA is not equal to AD.

1. A(B+C)=AB+AC
2. (A+B)C=AC+BC
3. (AB)C=A(BC)=ABC
4. λ (AB)= (λA)B=A(λB)

for appropriate order matrices A, B and C and scalar λ (a real number).

Let A be a square nxn matrix. Then the I,j **minor (ελάσων)** is the (n-1)x(n-1) resulting matrix after excluding the i^{th} row and the j^{th} column of A, e.g. for

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$$

the 2,2 minor M_{22} is

$$M_{22} = \begin{pmatrix} a_{11} & a_{13} & ... & a_{1n} \\ a_{31} & a_{32} & ... & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}.$$

The **Determinant (Ορίζουσα)** of a square nxn matrix A, | A| or det(A), is defined as

$$A = \sum_{i=1}^{n} a_{ij} \mid C_{ij} \mid, \text{ for } j = 1,2,...,\text{n} \text{ or } \sum_{j=1}^{n} a_{ij} \mid C_{ij} \mid, \text{ for } i = 1,2,...,\text{n}$$

where $|C_{ij}|$ are the I,j **cofactor (προσημασμένη ελάσσων)** of A, i.e.

$$|C_{ij}| = (-1)^{i+j} \, |M_{ij}|,$$

and $|M_{ij}|$ is the determinant of the I,j minor M_{ij}.
If |A|=0 then A is called **singular**. Otherwise it is called **non-singular**.

A square nxn matrix A is called **idempotent (ταυτοδύναμη)** iff

$$A^k = A \text{ for } k = 1,2,...$$

It suffices that

$$A^2 = A.$$

We find the **Transpose (Ανάστροφη)** of an nxk matrix A, say $A^{/}$ or A^T, by interchanging the row and columns of the matrix. Hence the order of matrix $A^{/}$ is kxn. If for a square matrix A we have $A=A^{/}$ then A is called **symmetric**.

The **Trace (Ίχνος)** of a square matrix nxn matrix A, tr(A), is the sum of the elements of the main diagonal, i.e.

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

Trace Properties

1. $tr(I_n) = n,$
2. $tr(A^{/}) = tr(A),$
3. $tr(AA^{/}) = tr(A^{/}A) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2,$
4. $tr(\lambda A) = \lambda tr(A),$
5. $tr(A+B) = tr(A) + tr(B),$
6. For appropriate matrices A, B and C $tr(ABC) = tr(CAB) = tr(BCA)$

Determinant Properties

1. $|I_n| = 1, \text{ and } |0_n| = 0,$
2. $|A| = |A^{/}|, \quad \text{and } |\lambda A| = \lambda^n |A|$
3. If every element of a row (or column) of a matrix is multiplied by λ then the determinant is multiplied by λ.
4. If two rows (or columns) of a matrix are equal or proportional then the determinant of this matrix is 0. Further, if a row (or column) of a matrix is 0 then the determinant is 0.
5. If B comes from the permutation of two rows (or columns) of A then $|B| = -|A|$
6. If A and B square matrices of the say order, say n, then
$$|AB| = |A||B| = |B||A| = |BA|$$
7. The determinant of a matrix does not change if in any row (or column) of the matrix we add any row (or column) multiplied by any real number.

8. If A diagonal, i.e. $A=\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}$ then $|A|=a_{11}a_{22}\dots a_{nn}$ The same is true if A is triangular.

9. A square nxn matrix A is called **orthogonal (ορθογώνια)** iff $A'A=AA'=\mathrm{I}_n$. Further for A orthogonal we have that $|A|=\pm 1$.

The **rank (βαθμός)** of an nxk matrix A, say r(A), is the maximum number of rows or columns that are linearly independent. If all rows (columns) are linearly independent then the matrix has **full row (column) rank**. Equivalently, the rank of a matrix is the order of the biggest non-singular submatrix of A.

Rank Properties

1. For an nxk matrix A $0\le r(A)\le \min(\mathrm{n},k)$
2. $r(I_n)=\mathrm{n}, \quad r(0_n)=0$
3. $r(A)=r(A')=r(AA')=r(A'A)$
4. If A and B matrices of the same order $r(A+B)\le r(A)+r(B), \text{ and } r(AB)\le \min[r(A),r(B)]$
5. If A diagonal $r(A)=n\text{umber of non-zero elements}$
6. If A idempotent, i.e. $A^2=A$ $r(A)=tr(A)$.
7. If A square nxn matrix then $|A|\ne 0 \Leftrightarrow r(A)=n$ and further $|A|=0\Leftrightarrow r(A)<n$.
8. If A square nxk and B kxl with r(B)=k then $r(AB)=r(A)$.

If A square non-singular matrix then there exist unique matrix B such that $BA=AB=I_n$. The matrix B is called the **inverse** of A and we write $B=A^{-1}$. A^{-1} is evaluated as $A^{-1}=\frac{1}{|A|}adjA$ where $adjA=C'$ and C is the matrix of cofactors of A.

Properties of Inverse

1. $I_n^{-1}=I_n$
2. $\left(A^{-1}\right)^{-1}=A$ and $\left(A'\right)^{-1}=\left(A^{-1}\right)'$
3. $|A^{-1}|=|A|^{-1}=\frac{1}{|A|}$
4. $\left(AB\right)^{-1}=B^{-1}A^{-1}$ and $\left(ABC\right)^{-1}=C^{-1}B^{-1}A^{-1}$

5. If A orthogonal $A^{-1} = A'$
6. If A non-singular and symmetric then A^{-1} non-singular symmetric.
7. If D diagonal non-singular with elements d_{ii}, then D^{-1} diagonal with elements $1/d_{ii}$

Two square nxn matrices A and B are called **Similar (Όμοιες)** if there exist non-singular matrix M such that $B = M^{-1}AM$

Properties of Similar matrices

1. $tr(A) = tr(B)$
2. $|A| = |B|$
3. $r(A) = r(B)$

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