ΚΑΛΗΣΠΕΡΑ

$$y_t = b_1 + b_2 x_{t2} + b_3 x_{t3} + \dots + b_k x_{tk} + \varepsilon_t \qquad \varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$$

$\Downarrow$

ε_t INDEPEN.

$E(\varepsilon_t) = 0 \ \forall t$

$V(\varepsilon_t) = \sigma^2 \ \forall t$

$t = 1, 2, \dots, n$

$$y_1 = b_1 + b_2 x_{12} + b_3 x_{13} + \dots + b_k x_{1k} + \varepsilon_1$$

$$y_2 = b_1 + b_2 x_{22} + b_3 x_{23} + \dots + b_k x_{2k} + \varepsilon_2$$

$$\vdots$$

$$y_n = b_1 + b_2 x_{n2} + b_3 x_{n3} + \dots + b_k x_{nk} + \varepsilon_n$$

$\Rightarrow$

$$y_1 = (1, x_{12}, x_{13}, \dots, x_{1k}) \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} + \varepsilon_1$$

$$\vdots$$

$$y_n = (1, x_{n2}, x_{n3}, \dots x_{nk}) \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} + \varepsilon_n$$

$\Rightarrow$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1k} \\ 1 & x_{22} & x_{23} & \cdots & x_{2k} \\ \vdots & & & & \\ 1 & x_{n2} & x_{n3} & \cdots & x_{nk} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\Rightarrow \underset{n\times 1}{Y} = \underset{n\times k}{X} \; \underset{k\times 1}{b} + \underset{n\times 1}{\varepsilon} \Rightarrow \varepsilon = Y - Xb$$

$$\min_b \sum_{t=1}^{n} \varepsilon_t^2 = \min_b (\varepsilon_1, \varepsilon_2, \dots \varepsilon_n) \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} =$$

$$= \min_b \varepsilon' \varepsilon = \min_b (Y - Xb)'(Y - Xb)$$

$$\Rightarrow \quad X'X\hat{\beta} = X'Y \Rightarrow \boxed{\hat{\beta} = (X'X)^{-1}X'Y}$$

LEAST SQUARED ESTIMATOR.

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + \varepsilon) = \cancel{(X'X)^{-1}X'X}\beta + (X'X)^{-1}X'\varepsilon$$

$$= \beta + (X'X)^{-1}X'\varepsilon.$$

$$E(\hat{\beta}) = E\left(\beta + (X'X)^{-1}X'\varepsilon\right) = \beta + E\left[(X'X)^{-1}X'\varepsilon\right] =$$

$$= \beta + (X'X)^{-1}X'E(\varepsilon) = \beta + (X'X)^{-1}X' \cdot \underset{n\times 1}{0} = \beta$$

$\hat{\beta}$ UNBIASED ESTIMATOR OF β.

$$E(\varepsilon) = E\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_n \end{pmatrix} = \begin{pmatrix} E(\varepsilon_1) \\ E(\varepsilon_2) \\ \vdots \\ E(\varepsilon_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underset{n\times 1}{0}$$

Z T.M. $V(Z) = E(Z - E(Z))^2$

$\underset{n \times 1}{Z}$ T. DIAN. $V(Z) = E\left[(Z - E(Z))(Z - E(Z))'\right]$

AN $E(Z) = 0$ $\Big\} \Rightarrow$

$$\underset{n\times n}{V(Z)} = E(\underset{n\times 1}{Z}\underset{1\times n}{Z'}) = E\left[\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}(z_1\ z_2 \dots z_n)\right]$$

$$= E\begin{pmatrix} z_1^2 & z_1 z_2 & \dots & z_1 z_n \\ z_2 z_1 & z_2^2 & \dots & z_2 z_n \\ z_n z_1 & z_n z_2 & \dots & z_n^2 \end{pmatrix} = \begin{pmatrix} E(z_1^2) & E(z_1 z_2) & \dots & E(z_1 z_n) \\ E(z_2 z_1) & E(z_2^2) & \dots & E(z_2 z_n) \\ \vdots & & & \\ E(z_n z_1) & E(z_n z_2) & \dots & E(z_n^2) \end{pmatrix}$$

$$= \begin{pmatrix} V(z_1) & Cov(z_1, z_2) & \dots & Cov(z_1, z_n) \\ Cov(z_2, z_1) & V(z_2) & \dots & Cov(z_2, z_n) \\ \vdots & & & \\ Cov(z_n, z_1) & Cov(z_n, z_2) & \dots & V(z_n) \end{pmatrix}$$

$n \times n$ SYMETRIC MATRIX

$$E(\varepsilon_t) = 0 \quad \forall t = 1, 2, \ldots, n \qquad \underset{n\times 1}{E(\varepsilon)} = \underset{n\times 1}{0}$$

$$V(\varepsilon_t) = \sigma^2 \quad \forall t = 1, 2, \ldots, n \qquad Cov(\varepsilon_t, \varepsilon_j) = 0 \quad \forall t \neq j = 1, \ldots, n$$

$$V(\varepsilon) = \begin{pmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & & \\ 0 & 0 & \ddots & & \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & \cdots & \sigma^2 \end{pmatrix} = \sigma^2 I_n$$

$$\underset{k\times 1}{V(\hat{\beta})} = E\left[(\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))'\right] =$$

$$= E\left[(X'X)^{-1}X'\varepsilon((X'X)^{-1}X'\varepsilon)'\right] =$$

$$= E\left((X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}\right) = (X'X)^{-1}X'E(\varepsilon\varepsilon')X(X'X)^{-1}$$

$$= \sigma^2 (X'X)^{-1}\underbrace{X'I_nX(X'X)^{-1}}_{I} = \sigma^2 (X'X)^{-1} \Longrightarrow$$

$$\underset{k\times k}{V(\hat{b})} = \begin{pmatrix} V(\hat{b}_1) & Cov(\hat{b}_1, \hat{b}_2) & \cdots & Cov(\hat{b}_1, \hat{b}_k) \\ Cov(\hat{b}_2, \hat{b}_1) & V(\hat{b}_2) & \cdots & Cov(\hat{b}_2, \hat{b}_k) \\ \vdots & & & \\ Cov(\hat{b}_k, \hat{b}_1) & - & & V(\hat{b}_k) \end{pmatrix} = \underbrace{\sigma^2 (X'X)^{-1}}$$

$\hat{\varepsilon} = Y - X\hat{b}$ RESIDUALS

$$S^2 = \frac{1}{n-k} \sum_{t=1}^{n} \hat{\varepsilon}_t^2 = \frac{1}{n-k} \hat{\varepsilon}' \hat{\varepsilon}$$ ESTIMATOR OF σ^2

S^2 THE LEAST SQUARES ETIMATOR OF σ^2

$$\widehat{V(\hat{b})} = S^2 (X'X)^{-1}$$

$$y_t = b_1 + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} (0, \sigma^2) \quad t = 1, 2, \dots n$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} b_1 + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$X'X = (1, 1, \dots, 1) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 1 + 1 + 1 + \dots + 1 = n$$

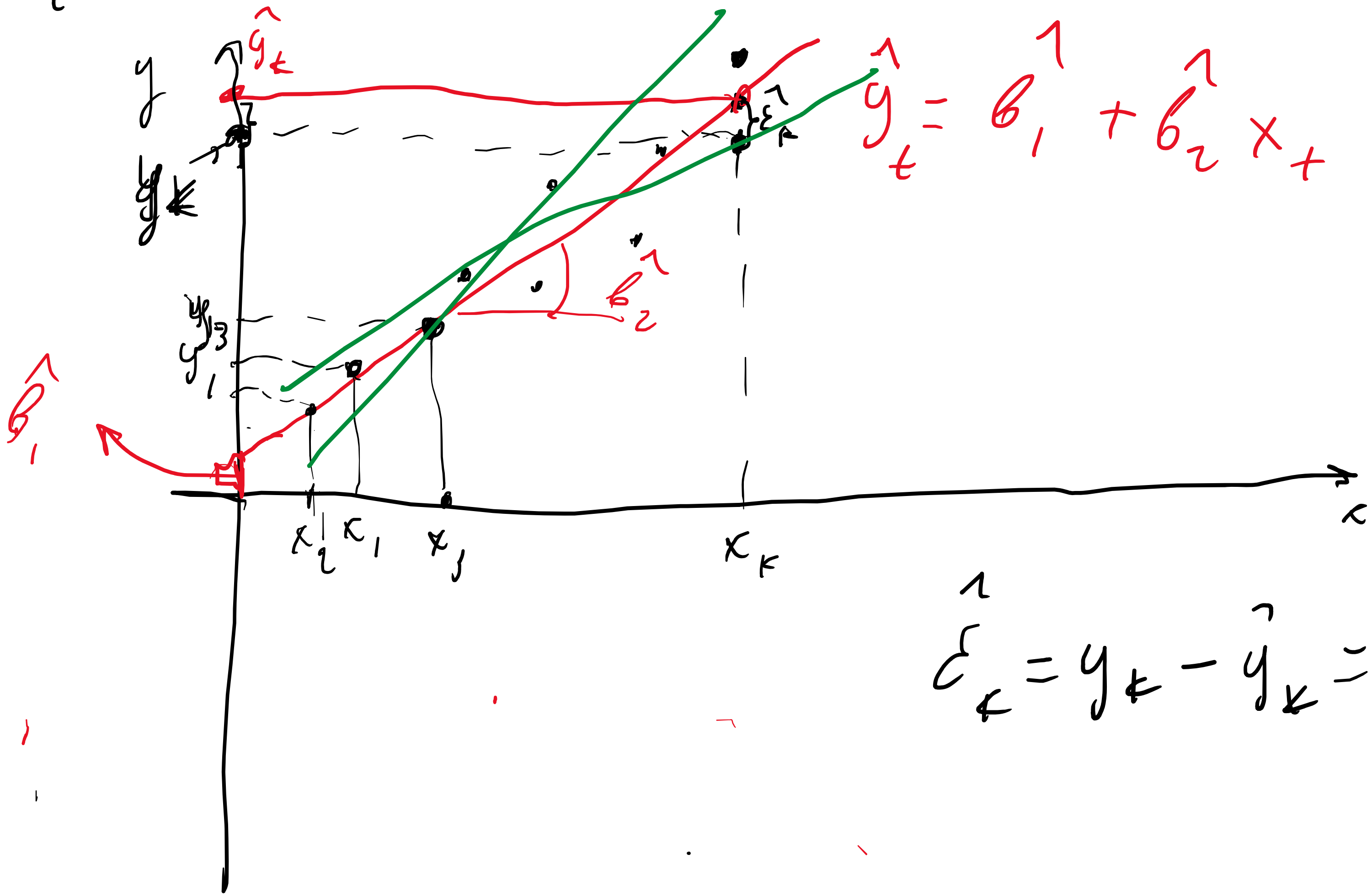
$$X'X = n$$

$$X'y = (1, 1, \dots 1) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = y_1 + y_2 + \dots + y_n$$

$$\hat{b}_1 = (X'X)^{-1} X'y = n^{-1}(y_1 + y_2 + \dots + y_n) = \frac{1}{n} \sum_{t=1}^{n} y_t =$$

$$= \bar{y}$$

$$y_t = b_1 + b_2 x_t + \varepsilon_t$$



$$\hat{y}_t = \hat{b}_1 + \hat{b}_2 x_t$$

$$\hat{\varepsilon}_k = y_k - \hat{y}_k = y_k - \hat{b}_1 - \hat{b}_2 x_k$$

$\hat{y} = X\hat{b}$ → FITTED VALUES

$\hat{\varepsilon} = y - \hat{y}$ → RESIDUAL

$\hat{y} = X\hat{b} = \underbrace{X(X'X)^{-1}X'}_{P}y = P \cdot y$

$P = X(X'X)^{-1}X'$ IDEMPOTENT (ΤΑΥΤΟΔΥΝΑΜΗ)

ΔΙΟΤΙ $P^2 = P \circ P = X\underbrace{(X'X)^{-1}X'X}(X'X)^{-1}X' = X\, I \cdot (X'X)^{-1}X'$

$= X(X'X)^{-1}X' = P$

P SYMMETRIC ΔΙΟΤΙ $P' = \left(X(X'X)^{-1}X'\right)' = (X')'\left((X'X)^{-1}\right)' \cdot X'$

$= X(X'X)^{-1}X' = P$

$$\text{Rank}(P) = r(P) \underset{\text{SYM.}}{\overset{\text{IDEMP.}}{=\!=}} tr(P) =$$

ΙΧΝΟΣ (TRACE)

$$= tr(X(X'X)^{-1}X') = tr((X'X)^{-1}X'X) = tr(I_k) = k.$$

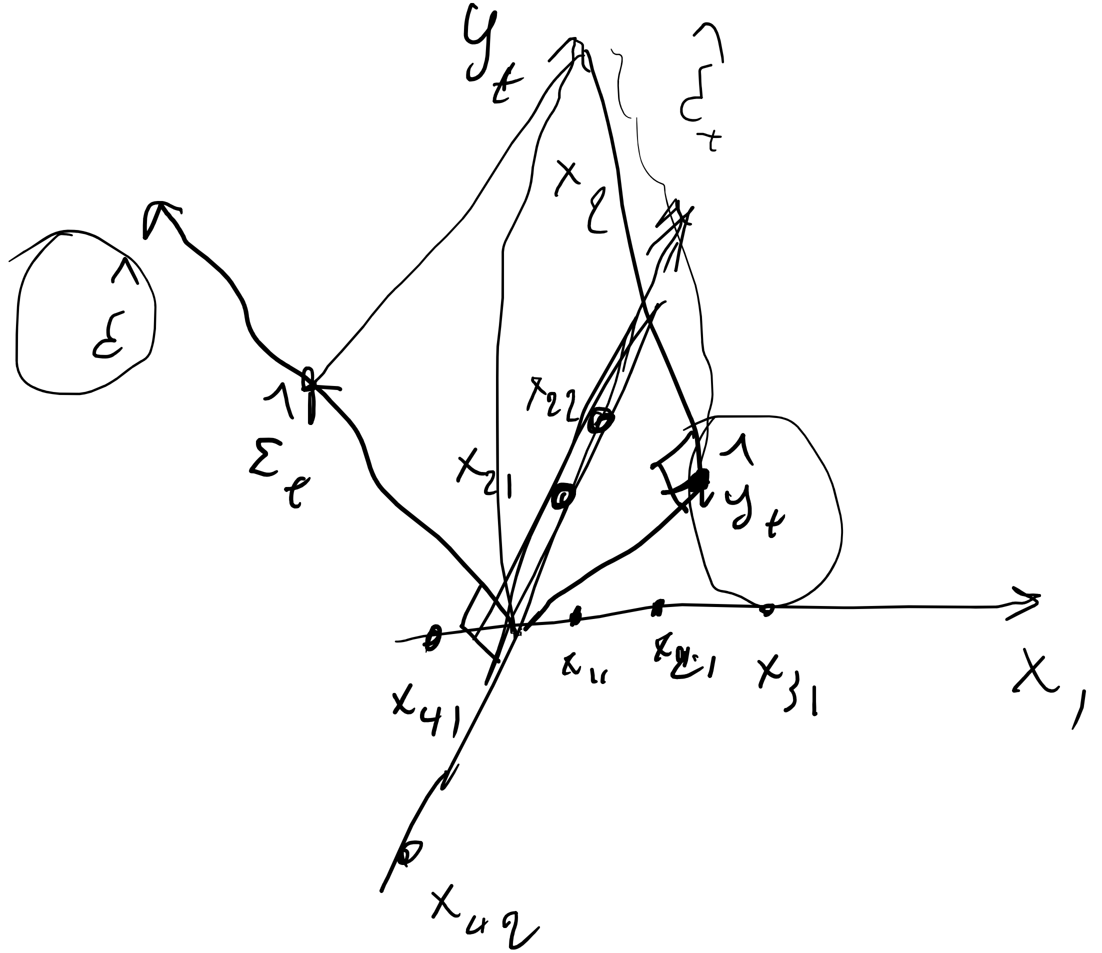
$r(\underset{n \times k}{X}) = k$ (ΥΠΟΘΕΣΗ) = ΑΡ. ΤΩΝ ΣΤΗΛΩΝ =

= ΑΡ. ΑΝΕΞΑΡΤΗΤΩΝ ΜΕΤΑΒΛΗΤΩΝ

$X_{t2} = 3X_{t1} + 5X_{t4}$ → ΓΡΑΜΜ. ~~ΕΞΑΡΤΗΜΕΝΕΣ~~

$P = X(X'X)^{-1}X'$ PROJECTION MATRIX (ORTHOGONAL) ON TO THE SPACE PRODUCED BY THE K INDEP. VARIA.

$$y_t = b_1 x_{1t} + b_2 x_{2t} + \varepsilon_t$$



$$\mathrm{corr}(x,y) = \frac{\mathrm{cov}(x,y)}{\sqrt{V(x)\cdot V(y)}}$$ CORRELATION COEFFICIENT.

$$\hat{\varepsilon} = y - \hat{y} = y - x\hat{b} = y - x(x'x)^{-1}x'y = \underbrace{[I_n - x(x'x)^{-1}x']}_{M = I_n - P} y$$

M IDEMPOTENT + SYMMETRIC

$$M^2 = [I_n - P][I_n - P] = I_n - P - P + P^2 = I_n - P - P + P$$
$$= I_n - P = M$$ IDEMP.

$$M' = [I_n - P]' = I_n - P' = I_n - P = M$$ SYM.

$$\hat{\varepsilon} = My, \quad r(M) = tr(M) = tr(I_n - P) = tr(I_n) - tr(P)$$
$$= n - k$$

$$Mx = [I_n - x(x'x)^{-1}x']x = x - x\underbrace{(x'x)^{-1}x'x}_{I_k} = x - x = 0$$

M PROJECTS ORTHOGONALLY Y ON TO A SPACE ORTHOGONAL TO X

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2) \quad t = 1, 2 \ldots, n$$

$$\underset{n\times1}{\varepsilon} \sim N\left(\underset{n\times1}{0}, \sigma^2 I_n\right)$$

$$\hat{b} = \underline{b} + \underbrace{(X'X)^{-1}X'}\underset{\uparrow N}{\bar{\varepsilon}} \Rightarrow \hat{b} \sim N\left(b, \sigma^2 (X'X)^{-1}\right)$$

$$\Rightarrow \hat{b} - b \sim N\left(0, \sigma^2 (X'X)^{-1}\right) \Rightarrow ?$$

$$\Rightarrow (\hat{b} - b)' \left[\sigma^2 (X'X)^{-1}\right]^{-1} (\hat{b} - b) \sim \chi^2_K$$

$$r(\underset{n \times k}{X}) = k \Rightarrow r(\underbrace{X'X}_{k \times k}) = k \Rightarrow |X'X| \neq 0 \Rightarrow \exists (X'X)^{-1}$$

$$(X'X)^{-1} = \frac{1}{|X'X|} \operatorname{Adj}(X'X)$$ ΤΕΡΑΣΤΙΟΙ ΑΡΙΘΜΟΙ

$|X'X| \sim 0$

$$\Rightarrow V(\hat{b}) = \sigma^2 (X'X)^{-1}$$ ΤΕΡΑΣΤΙΟΙ

$$\hat{b} \sim N(b, \sigma^2 (X'X)^{-1})$$

$$\hat{b}_i \sim N(b_i, \sigma^2 x^{ii})$$ ΟΠΟΥ x^{ii} ΤΟ (i,i) ΣΤΟΙΧΕΙΟ ΤΗΣ $(X'X)^{-1}$

$$\Rightarrow \frac{\hat{b}_i - b_i}{\sqrt{\sigma^2 x^{ii}}} \sim N(0,1)$$

$$s^2 = \frac{1}{n-k} \sum_{t=1}^{n} \hat{\varepsilon}_t^2$$

$$\underline{\hat{\varepsilon}} = y - X\hat{b} = (I - X(X'X)^{-1}X')y$$

$$= My = M(Xb+\varepsilon) =$$

$$= \underbrace{MX}_{0} b + M\varepsilon = \underline{M\varepsilon}$$

$$E(s^2) = \sigma^2$$

$$\hat{\varepsilon} \sim N(0, \sigma^2 M) \Rightarrow \hat{\varepsilon}'\hat{\varepsilon} \sim \sigma^2 \chi^2_{n-k}$$

$$\uparrow$$

$$\Rightarrow \frac{1}{\sigma^2} \sum_{t=1}^{n} \hat{\varepsilon}_t^2 \sim \chi^2_{n-k}$$

$$\frac{\hat{b}_i - b_i}{\sqrt{\sigma^2 x^{ii}}} \sim N(0,1)$$

$$\Rightarrow \frac{\dfrac{\hat{b}_i - b_i}{\sqrt{\sigma^2 x^{ii}}}}{\sqrt{\dfrac{1}{\sigma^2} \sum_{t=1}^{n} \hat{\varepsilon}_t^2 \dfrac{1}{n-k}}} \sim t_{n-k}$$

s^2

$$\Rightarrow \frac{\hat{b}_i - b_i}{\sqrt{s^2 x^{ii}}} \sim t_{n-k}$$

$H_0: b_i = k$ v.s. H_1 $b_i \neq k$

$$\frac{\hat{b}_i - k}{\sqrt{s^2 x^{ii}}} \overset{H_0}{\sim} t_{n-k} \Rightarrow \frac{\hat{b}_i - k}{s.e.(\hat{b}_i)} \overset{H_0}{\sim} t_{n-k}$$

$H_0: b_i = 0$ v.s. $H_1: b_i \neq 0$

$$t = \frac{\hat{b}_i}{s.e.(\hat{b}_i)} \overset{H_0}{\sim} t_{n-k}$$

Reject H_0 iff

$|t| > t_{n-k, \alpha/2} \iff$

Reject H_0 iff

$P.V.(t) < \alpha$

coeff		s.e	t-stat.	P.V.		
$\hat{\beta}_1$	0.5	0.1	5	0.00	$\rightarrow$ Reject H_0	$\beta_1 = 0$
$\hat{\beta}_2$	-0.3	0.3	-1	0.15	$\rightarrow$ DONOT REJ.	$\beta_2 = 0$
$\hat{\beta}_3$	7	0.2	-35	0.00	$\rightarrow$ Reject	$\beta_3 = 0$

b_1, b_3 STATISTICALLY SIGNIFICANT

b_2 >> INSIGNIFICANT