$$Y = X\beta + \varepsilon, \quad \varepsilon_i \sim (0, \sigma_i^2) \quad \forall i = 1, 2, \ldots$$

$$Cov(\varepsilon_i, \varepsilon_j) = 0 \quad \forall\, i \neq j = 1, 2, \ldots, n$$

$$V(\varepsilon) = \begin{pmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ 0 & & & \ddots \ \sigma_n^2 \end{pmatrix}$$

$\hat{\beta} = (X'X)^{-1} X'Y \rightarrow$ UNBIASED BUT INEFFICIENT.

$$V(\hat{\beta}) = (X'X)^{-1} \underbrace{X' \Omega X}_{} (X'X)^{-1}$$

$$\underline{\Omega}^{-1} = \begin{pmatrix} 1/\sigma_1^2 & & & 0 \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ 0 & & & 1/\sigma_n^2 \end{pmatrix}$$

$$\underline{\Omega}^{-1} = PP'$$

$$P = \begin{pmatrix} 1/\sigma_1 & & & 0 \\ & 1/\sigma_2 & & \\ & & \ddots & \\ 0 & & & 1/\sigma_n \end{pmatrix}$$

$$PY = PXb + \underbrace{P\varepsilon}$$

$$PY = \begin{pmatrix} 1/\sigma_1 & & & 0 \\ & 1/\sigma_2 & & \\ & & \ddots & \\ 0 & & & 1/\sigma_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} =$$

$$= \begin{pmatrix} y_1/\sigma_1 \\ y_2/\sigma_2 \\ \vdots \\ y_n/\sigma_n \end{pmatrix} \qquad X = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_k \\ | & | & & | \end{pmatrix}$$

$$X_1^* = \begin{pmatrix} x_{11}/\sigma_1 \\ x_{21}/\sigma_2 \\ \vdots \\ x_{n1}/\sigma_n \end{pmatrix}, \quad X_2^* = \begin{pmatrix} x_{12}/\sigma_1 \\ x_{22}/\sigma_2 \\ \vdots \\ x_{n2}/\sigma_n \end{pmatrix}$$

$$\cdots \quad X_t^* = \begin{pmatrix} x_{1t}/\sigma_1 \\ x_{2t}/\sigma_2 \\ \vdots \\ x_{nt}/\sigma_n \end{pmatrix}$$

WHITE

$$\hat{\sigma}_i^2 = \hat{\varepsilon}_i^2 \quad , \quad \hat{\varepsilon}_i = y_i - x_i \hat{\beta}$$

$$x_i \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\hat{\mu} = \frac{1}{n} \sum x_i \quad , \quad \hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$$

$$\hat{b} = (X'X)^{-1} X'y$$

$$V(\hat{b}) = (X'X)^{-1} X' \hat{\underline{\Omega}} X (X'X)^{-1}$$

$$\hat{\underline{\Omega}} = \begin{pmatrix} 1/\hat{\varepsilon}_1^2 & & & 0 \\ & 1/\hat{\varepsilon}_2^2 & & \\ & & \ddots & \\ 0 & & & \frac{1}{\hat{\varepsilon}_n^2} \end{pmatrix}$$

$\hat{b}$ IS ROBUST TO HETEROSKEDASTICITY OF UNKNOWN FORM.

$$y_t = x_t' b + \varepsilon_t, \quad \varepsilon_t | I_{t-1} \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2$$

Autoregressive CONDITIONALLY HETEROSKEDASTIC of order 1

ARCH(1)

$$I_{t-1} = \{ y_{t-1}, x_{t-1}, \ldots \}$$

$$\sim \{ \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots \}$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 \Rightarrow \varepsilon_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \underbrace{\varepsilon_t^2 - \sigma_t^2}_{v_t}$$

$$\Rightarrow \varepsilon_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + v_t$$

AR(1) 6Tq ε_t^2 <u>ANAN</u>

$E(v_t) = 0$

$E(v_t v_{t-k}) = 0$

$E(\varepsilon_{t-1}^2 v_t) = 0$

$$E(v_t) = E\left[E(v_t \mid I_{t-1})\right] =$$

$$= E\left[E(\varepsilon_t^2 - \sigma_t^2) \mid I_{t-1}\right] =$$

$$= E\left[E(\varepsilon_t^2 \mid I_{t-1}) - \sigma_t^2\right] = E(\sigma_t^2 - \sigma_t^2) = 0$$

For $|\alpha| < 1 \Rightarrow E(\varepsilon_t^2) = w + \alpha E(\varepsilon_{t-1}^2) + E(v_t)$

$\Rightarrow E(\varepsilon_t^2)(1-\alpha) = w \Rightarrow E(\varepsilon_t^2) = \frac{w}{1-\alpha}$

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2$$

POSITIVITY CONSTRAINTS

CONSTRAINTS ON THE Parameters of the COND. VAR. EQUATION such that

$$P(\sigma_t^2 > 0) = 1 \Leftrightarrow P(\sigma_t^2 < 0) = 0$$

FOR ARCH(1) $w > 0$, $\alpha \geq 0$

For σ_t^2 with POSIT. CONSTR + STATION.

$\Rightarrow 0 \leq \alpha < 1$

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2$$

$$\sigma_t^2 = w + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \dots + \alpha_q \varepsilon_{t-q}^2$$

ARCH(q)

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \qquad \text{GARCH}(1,1)$$

$$\Leftrightarrow \varepsilon_t^2 = w + (\alpha+\beta)\varepsilon_{t-1}^2 + V_t - \beta V_{t-1}$$

ARMA (1, 1)

ε_t^2 stationary iff $(\alpha+\beta) < 1$

$$\mathrm{Corr}\left(\varepsilon_t^2, \varepsilon_{t-k}^2\right) = (\alpha+\beta)^{k-1}\rho_1$$

$$\rho_1 = \mathrm{Corr}(\varepsilon_t^2, \varepsilon_{t-1}^2)$$

$\alpha+\beta$ = PERSISTANCE

POSITIVITY CONSTRAINTS

$\omega > 0, \quad \alpha \geq 0, \quad \beta \geq 0$

GARCH (2, 1)

$$\sigma_t^2 = \omega + b_1 \sigma_{t-1}^2 + b_2 \sigma_{t-2}^2 + \alpha \varepsilon_{t-1}^2$$

GARCH (1, 2)

$$\sigma_t^2 = \omega + b \sigma_{t-1}^2 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2$$

GARCH (1, 1)

POSIT. CONST + station $\Rightarrow$

$\omega > 0$

$\alpha \geq 0 \quad 0 \leq \alpha + b < 1$

$\beta \geq 0$