# STRUCTURAL STABILITY TESTS

One MAIN assumption of the classical regression is that the parameters, $\beta$, are the same for the whole sample, i.e. for $t = 1, 2, 3, \ldots, n$

$$y_t = \beta_1 + x_{t2}\beta_2 + \cdots + x_{tk}\beta_k + \varepsilon_t \quad \text{OR}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Assume now that this is not true, i.e.

For the first $n_1$ observations we have

① $Y_1 = X_1 \beta^* + \varepsilon^*_{n_1}$ and for the last $n_2$ observations

② $Y_2 = X_2 \beta^{**} + \varepsilon^{**}_{n_2}$

UNDER $H_0: \beta = \beta^* = \beta^{**}$ we get $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \beta + \begin{pmatrix} \varepsilon^* \\ \varepsilon^{**} \end{pmatrix} \Rightarrow$

$\Rightarrow$ ③ $Y = X\beta + \varepsilon$ for $n_1 + n_2 = n$. NOW

Let $\hat{\varepsilon}^* = Y_1 - X_1 \hat{\beta}^*$ and $\hat{\varepsilon}^{**} = Y_2 - X_2 \hat{\beta}^{**}$

and $\hat{\varepsilon} = Y - X\hat{\beta}$. NOW

$\hat{\varepsilon}^*$ and $\hat{\varepsilon}^{**}$ are the UNRESTRICTED RESIDUALS

whereas $\hat{\varepsilon}$ are the RESTRICTED ONES

Hence

$$F = \frac{\left(\hat{\varepsilon}'\hat{\varepsilon} - \hat{\varepsilon}^{*\prime}\hat{\varepsilon}^{*} - \hat{\varepsilon}^{**\prime}\hat{\varepsilon}^{**}\right)/k}{\dfrac{\hat{\varepsilon}^{*\prime}\hat{\varepsilon}^{*} + \hat{\varepsilon}^{**\prime}\hat{\varepsilon}^{**}}{n-2k}} \overset{H_0}{\sim} F(k, n-2k)$$

which is known as the CHOW test.

# RAMSEY RESET test

$H_0: \varepsilon \sim N(0, \sigma^2 I_n)$

$H_1: \varepsilon \sim N(\mu, \sigma^2 I_n)$

$$Y = X\beta + \varepsilon \longrightarrow Y = X\beta + Z\alpha + \varepsilon$$

$n \times 1 \quad n \times k \quad n \times r \quad r \times 1 \quad n \times 1$

$$\hat{\beta} = (X'X)^{-1}X'Y \longrightarrow \hat{Y} = X\hat{\beta}$$

$$Z = \{\hat{y}_i^2, \hat{y}_i^3, \hat{y}_i^4\}_{i=1,\dots,n}$$

$H_0: \alpha = 0$ v.s. $H_1: \alpha \neq 0$

USUAL F

$$\hat{b} = (X'X)^{-1}X'Y \rightarrow \hat{\varepsilon} = Y - X\hat{b}$$

$$\begin{pmatrix}\hat{b}\\ \hat{d}\end{pmatrix} = \left((X' Z') \begin{pmatrix} X \\ Z \end{pmatrix}\right)^{-1} (X' Y') Y$$

$$Z = (\hat{Y}^2, \hat{Y}^3, \hat{Y}^4)$$

$$\Rightarrow \hat{e} = Y - \begin{pmatrix} X \\ Z \end{pmatrix}' \begin{pmatrix}\hat{b}\\ \hat{d}\end{pmatrix}$$

$$F = \frac{(\hat{\varepsilon}'\hat{\varepsilon} - \hat{e}'\hat{e}) / 3}{\hat{e}'\hat{e} / (n-k-3)} \overset{H_0}{\sim} F(3, n-k-3).$$

Let $\tau$ date of structured Break.

$$\tau_1 = n\alpha \%_0 \qquad \tau_2 = n(1-\alpha \%_0)$$

F statistics Chow

$n_1 = \tau_1, \quad n_2 = \tau_2$

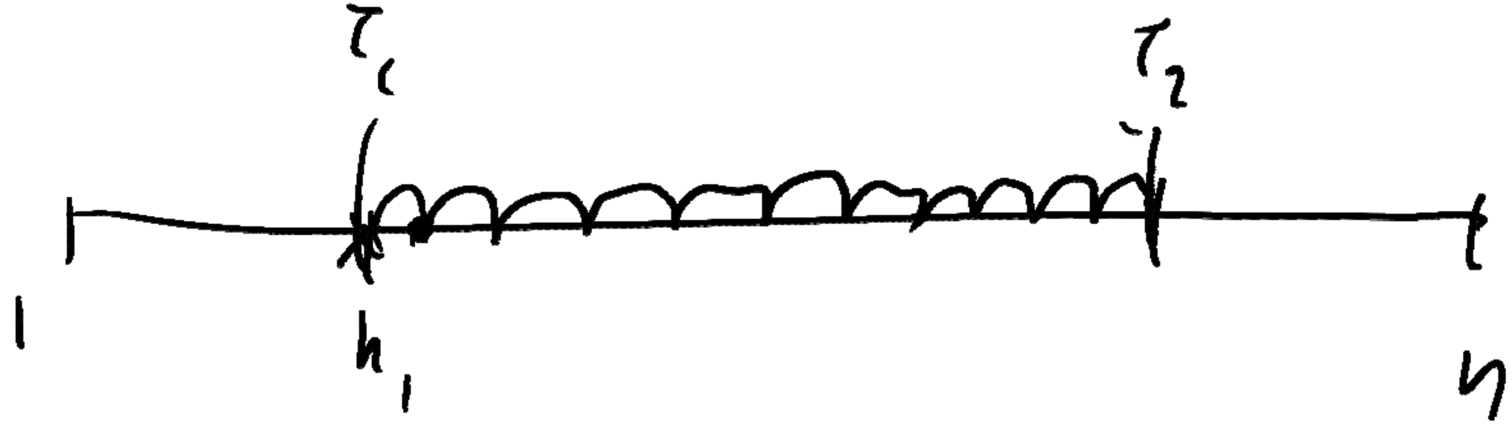
NON STANDARD CRIT. VAL.

$$\max F = \max_{\tau_1 \le \tau \le \tau_2} F(\tau)$$

$$\text{EXP } F = \ln\left[\frac{1}{K} \sum_{\tau=\tau_1}^{\tau_2} \exp\left(\frac{1}{2} F(\tau)\right)\right]$$

$$\text{AVE } F = \frac{1}{K} \sum_{\tau=\tau_1}^{\tau_2} F(\tau)$$

Ecola 1993 Andrews.




$$\text{MAX}\ \{ F(\tau_1),\ F(\tau_1+1),\ F(\tau_1+2) \ldots$$

$$\underbrace{F(\tau_1+\lambda)} \ldots F(\tau_2) \} \rightarrow \text{critical Andrews.}$$

$\underline{\text{MAX}}$ $\tau_1+\lambda$ ExD BREAK.

QUANDT-ANDREWS

$C_t = \alpha + b y_t + \varepsilon_t$ $y_t$ στατ. μεταβλητός

$t$ (χρονιά πολέμου)

$\alpha^* < \alpha$

DUMMY VARIABLE $D_t = \begin{cases} 0 & \text{χρόνια ειρήνης} \\ 1 & \text{χρόνια πολέμου} \end{cases}$

$C_t = \alpha + \delta \cdot D_t + b y_t + \varepsilon_t$

$\delta < 0$ στατιστικά σημαντικό $D_t$

$t$ πολέμου $C_t = \alpha + \delta + b y_t + \varepsilon_t$

$t$ χρονιά ειρήνης $C_t = \alpha + b y_t + \varepsilon_t$.

$$C_t = \alpha + \delta D_t + b\, y_t + \gamma \underbrace{y_t \cdot D_t}_{Z_t} + \varepsilon_t$$

Πόλεμο $(D_t = 1)$

$$C_t = \alpha + \delta + (b + \gamma)\, y_t + \varepsilon_t$$

ΕΙΡΗΝΗ $(D_t = 0)$

$$C_t = \alpha + b\, y_t + \varepsilon_t$$

$$D_t = \begin{cases} 0 & \text{ΕΙΡ.} \\ 1 & \text{ΠΟΛ.} \end{cases} \quad , \quad D_t^* = \begin{cases} 1 & \text{ΕΙΡ} \\ 0 & \text{ΠΟΛ.} \end{cases}$$

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$C_t = \cancel{\alpha} + \delta D_t + \delta^* D_t^* + b\, y_t + \varepsilon_t$$

$$r_t = -0.05 + 3.2\, D_t + 0.82\, rm_t - \cancel{0.01\, rm_t \cdot D_t}$$
$$\quad (0.001) \quad (0.00) \quad (0.000) \quad \cancel{(0.18)}$$

P.V. $\rightarrow$

$$+ 1.2\, sm_t + 0.25\, hm_t - 0.24\, hm_t \cdot D_t$$
$$(0.00) \quad (0.00) \quad (0.00)$$

$rm_\tau = 10\%$, $sm_\tau = 5\%$, $hm_\tau = -10\%$

$\tau$ JUN.

$$\hat{r}_\tau = -0.05 + 3.2 + 0.82\,(10\%) + 1.2\,(5\%) + 0.25\,(-10\%) - 0.24\,(-10\%)$$

$\tau$ NON-JUN.

$$\hat{\tilde{r}}_\tau = -0.05 + 0.82\,(10\%) + 1.2 \cdot (5\%) + 0.25 \cdot (-10\%)$$

For our data the Andrews - Quandt test shows a Break at January 1973

Hence we introduce a JANUARY DUMMY, i.e.

$$JANDUM_t = \begin{cases} 1 & \text{for month = JANUARY} \\ 0 & \text{otherwise.} \end{cases}$$

# MAXIMUM LIKELIHOOD Method

$$\underset{n\times 1}{Y = X\beta + \varepsilon} \longrightarrow \hat{\beta} = (X'X)^{-1}X'Y$$

$\hat{\beta}$ MINIMIZES $\varepsilon'\varepsilon \Rightarrow \min_{\beta} (Y-X\beta)'(Y-X\beta)$

$\hat{\beta}_{OLS}$. NOTICE THAT WE DONOT ASSUME ANY DISTRIBUTION FOR THE ERROR, $\varepsilon$.

ASSUME NOW That $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

THEN

$$y_t = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + \cdots + \beta_K x_{tK} + \varepsilon_t$$

We have

$$E(y_t) = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + \cdots + \beta_K x_{tK}$$

and

$$V(y_t) = V(\varepsilon_t) = \sigma^2$$

ALSO as $\varepsilon_t \overset{iid}{\sim} N( ) \Rightarrow y_t \sim N( \quad )$ INDEPEND.

Hence

$$y_t \sim N(\beta_1 + \beta_2 x_{t2} + \cdots + \beta_K x_{tK}, \sigma^2)$$

INDEPENDENTLY.

Hence

$$f(y_t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(y_t - b_1 - b_2 x_{t2} - \dots - b_k x_{tk}\right)^2}$$

Now the LIKELIHOOD is given as

$$f(y_1, y_2, \dots, y_n) = \prod_{t=1}^{n} f(y_t)$$

and the log-LIKELIHOOD as

$$\ell(b_1, b_2, \dots, b_k, \sigma^2) = \sum_{t=1}^{n} \ln f(y_t) =$$

$$= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{n}(y_t - \beta_1 - \cdots - \beta_k x_{tk})^2$$

$$= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)$$

NOTICE

$$\max_{\beta} \ell(\beta_1, \ldots, \beta_k, \sigma^2) = \max_{\beta} -\frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)$$

$$= \min_{\beta} \frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta) = \min_{\beta}(Y - X\beta)'(Y - X\beta)$$

Hence $\hat{\beta}_{MLS} = \hat{\beta}_{OLS} = (X'X)^{-1}X'Y$

However this is not the case for $\sigma^2$

$$\frac{\partial l}{\partial \sigma^2} = 0 \Rightarrow -n \frac{1}{\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}(Y - X\hat{\beta})'(Y - X\hat{\beta}) = 0$$

$$\Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n} (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

$$= \frac{1}{n} \hat{\varepsilon}' \hat{\varepsilon} = \frac{1}{n} \sum_{i=1}^{n} \hat{\varepsilon}_i^2$$

$$S^2_{OLS} = \frac{1}{n-K} \sum_{i=1}^{n} \hat{\varepsilon}_i^2 \neq \hat{\sigma}^2_{MLE}$$

$$Y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2 \Omega)$$

$\Omega$ ΘΕΤΙΚΑ ΟΡΙΣΜΕΝΗ ΣΥΜΜΕΤΡΙΚΗ ΜΗΤΡΑ

$$V(\varepsilon_i) = \sigma^2 w_{ii}, \quad Cov(\varepsilon_i, \varepsilon_j) = \sigma^2 w_{ij}$$

$V(\varepsilon_i) = \sigma^2 \ \forall i$ ΔΕΝ ΙΣΧΥΕΙ

$Cov(\varepsilon_i, \varepsilon_j) = 0 \ \forall i \neq j$ ΔΕΝ ΙΣΧΥΕΙ

$$E(Y) = X\beta + \underbrace{E(\varepsilon)}_{0} = X\beta$$

$$V(Y) = E\left[(Y - E(Y))(Y - E(Y))'\right] =$$

$$= E(\varepsilon\varepsilon') = \sigma^2\Omega$$

$$\boxed{Y \sim N(X\beta, \sigma^2\Omega)}$$

$$\ell(\beta, \sigma^2, \Omega \mid Y) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2}\ln|\Omega| - \frac{1}{2\sigma^2}(Y - X\beta)'\Omega^{-1}(Y - X\beta) \leftarrow$$

$$\frac{\partial l}{\partial \beta} = 0 \Rightarrow \frac{1}{\sigma^2}\left(X'\Omega^{-1}y - X'\Omega^{-1}X\hat{\beta}\right) = 0$$

$$\Rightarrow \hat{\beta}_{MLE} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

$$\frac{\partial l}{\partial \sigma^2} = 0 \Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

$\underline{\Omega}$ ΘΕΤΙΚΑ ΟΡΙΣΜΕΝΗ ΜΗΤΡΑ $\Rightarrow$

$\underline{\Omega}^{-1}$ » » » » $\Rightarrow$

$$\exists\ P : |P| \neq 0 : \underline{\Omega}^{-1} = P'P$$

$$\hat{\beta}_{MLE} = (X'\underline{\Omega}^{-1}X)^{-1} X'\underline{\Omega}^{-1}y =$$

$$= (X'P'PX)^{-1} X'P'Py =$$

$$= [(PX)'PX]^{-1}(PX)'Py =$$

$= \hat{b}_{OLS}$ THE ΠΑΛΙΝΔΡΟΜΗΣΗΣ

$PY$ ON $PX$

$$PY = PX\, b + P\varepsilon$$

$$\hat{b}_{OLS} = \left((PX)'PX\right)^{-1} \cdot (PX)'PY$$

$$Y^* = X^* b + \varepsilon^*$$

$$\hat{b}_{OLS} = (X^{*\prime}X^*)^{-1}(X^*)'Y^* \rightarrow \text{BLUE}$$

iff $V(\varepsilon^*) = \sigma^2 I_n$

$$V(\varepsilon^*) = V[P\varepsilon] = P V(\varepsilon) P' =$$

$$= P \sigma^2 \Omega P' = \sigma^2 P (P'P)^{-1} P' =$$

$$= \sigma^2 \underbrace{P P^{-1}}_{I} \underbrace{(P')^{-1} P'}_{I} = \sigma^2 I_n .$$

$$V(\varepsilon) = \sigma^2 \Omega \neq \sigma^2 I_n$$

$$\hat{\beta}_{MLE} = (X'\Omega^{-1}X)^{-1} X'\Omega^{-1} Y = \hat{\beta}_{GLS}$$

GLS = GENERALISED LEAST SQUARES

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad \varepsilon \sim (0, \sigma^2 \underline{\Omega})$$

ASSUME WRONGLY $\varepsilon \sim (0, \sigma^2 I_n)$

$$\hat{\beta} = (X'X)^{-1}X'Y = \beta + (X'X)^{-1}X'\varepsilon$$

$$E(\hat{\beta}) = \beta + (X'X)^{-1}X'E(\varepsilon) = \beta$$

$$V(\hat{\beta}) = E\left[\left[(X'X)^{-1}X'\varepsilon\right]\left[(X'X)^{-1}X'\varepsilon\right]'\right]$$

$$= (X'X)^{-1}X'\sigma^2\underline{\Omega}\,X(X'X)^{-1} \neq \sigma^2(X'X)^{-1}$$

UNLESS $\underline{\Omega} = I_n$

$$V(\hat{b}) = E\left[(\hat{b} - E(\hat{b}))(\hat{b} - E(\hat{b}))'\right]$$

$$= E\left[(\hat{b} - b)(\hat{b} - b)'\right] =$$

$$= E\left[(X'X)^{-1}X'\varepsilon\left[(X'X)^{-1}X'\varepsilon\right]'\right] =$$

$$= E\left[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}\right] =$$

$$= (X'X)^{-1}X'E(\varepsilon\varepsilon')X(X'X)^{-1} =$$

$$= \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}$$

ECONOMETRIC PACKAGES EVALUATE

$$\hat{V}(\hat{b}) = s^2 (X'X)^{-1}$$

← ΛΑΘΟΣ

$$s^2 = \frac{1}{n-k} \sum_{t=1}^{n} \hat{\varepsilon}_t^2$$

$$V(\hat{b}) = \sigma^2 (X'X)^{-1} X' \Omega X (X'X)^{-1}$$

⇒ standard errors of $\hat{b}_i$, $i = 1, 2, \ldots, k$

ARE WRONG ⇒

⇒ WRONG INFERENCE.