$Cov(\varepsilon_i, \varepsilon_j) = 0 \ \forall \ i \neq j$

AUTOCORRELATION → AR AUTOREGRESSIVE
→ MA MOVING AVERAGE
→ ARMA

AR(K)
↑ K IS THE ORDER OF AR MODEL

$\varepsilon_t = \mu + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \dots + \varphi_K \varepsilon_{t-k} + n_t$

AR(1) $\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + n_t$

$n_t \overset{iid}{\sim} (0, \sigma^2)$

STRONG STATIONARITY

$\ldots, y_{t-1}, y_t, y_{t+1}, \ldots$ IS STRONGLY STATIONARY iff

$f(y_{t+1}, y_{t+2}, \ldots, y_{t+k})$ is the same

$\forall t, \forall k$.

WEAK STATIONARITY OF ORDER ν

$\ldots, y_1, y_2, \ldots, y_n, \ldots$ is weakly stationary of ORDER ν iff $E(y_t)$, $E(y_t^2)$, $E(y_t^3)$, $\ldots$, $E(y_t^\nu)$ is independent of t

y_t is 2nd ORDER stationary iff

$E(y_t)$, $E(y_t^2)$, $E(y_t y_{t-k})$ independent of t

---

y_t 2nd ORDER STAT. + $y_t \sim N(\cdot,\cdot)$ $\Rightarrow$ y_t STRONGLY STATION.

---

y_t STRONGLY STA + $V(y_t) < \infty$ $\Rightarrow$ y_t 2nd ORDER STATION.

AR(1)

$$\varepsilon_t = \mu + \varphi\, \varepsilon_{t-1} + u_t \qquad u_t \overset{iid}{\sim} (\underset{=}{0}, \sigma^2)$$

$$E(\varepsilon_t) = \mu + \varphi E(\varepsilon_{t-1}) + \underbrace{E(u_t)}_{=0}$$

$$\Rightarrow E(\varepsilon_t) = \mu + \varphi E(\varepsilon_{t-1})$$

if STATION.

$$\Rightarrow E(\varepsilon_t)(1-\varphi) = \mu \Rightarrow E(\varepsilon_t) = \frac{\mu}{1-\varphi}$$

$$V(\varepsilon_t) = V(\mu + \varphi \varepsilon_{t-1} + u_t) =$$

$$\Rightarrow V(\varepsilon_t) = V(\varphi \varepsilon_{t-1} + u_t) \Rightarrow$$

$$\Rightarrow V(\varepsilon_t) = \varphi^2 V(\varepsilon_{t-1}) + V(u_t) + 2\varphi\, Cov(\varepsilon_{t-1}, u_t)$$

$$\Rightarrow V(\varepsilon_t) = \varphi^2 V(\varepsilon_{t-1}) + \sigma^2 + 0$$

if STATION. $\Rightarrow$ $V(\varepsilon_t) = \dfrac{\sigma^2}{1-\varphi^2}$

$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + u_t =$

$= \mu + \varphi\mu + \varphi^2 \varepsilon_{t-2} + u_t + \varphi u_{t-1}$

$= \mu + \varphi\mu + \varphi^2 \mu + \varphi^3 \varepsilon_{t-3} + u_t + \varphi u_{t-1} + \varphi^2 \varepsilon_{t-2}$

$= \mu(1+\varphi+\varphi^2+\dots) + u_t + \varphi u_{t-1} + \varphi^2 u_{t-2} + \varphi^3 u_{t-3} + \dots$

$\Rightarrow E(\varepsilon_t) = \mu(1+\varphi+\varphi^2+\varphi^3+\dots)$

$\Rightarrow E(\varepsilon_t) = \dfrac{\mu}{1-\varphi}$ iff $|\varphi| < 1$ 4

$$V(\varepsilon_t) = V\left(k(1+\phi+\phi^2+\dots) + u_t + \phi u_{t-1} + \phi^2 u_{t-2} + \dots\right)$$

$$= V\left(u_t + \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \dots\right)$$

$$= V(u_t) + \phi^2 V(u_{t-1}) + \phi^4 V(u_{t-2}) + \phi^6 V(u_{t-3}) + \dots$$

$$\Rightarrow V(\varepsilon_t) = \sigma^2 (1 + \phi^2 + \phi^4 + \phi^6 + \dots)$$

$$\Rightarrow V(\varepsilon_t) = \frac{\sigma^2}{1-\phi^2} \quad \underline{\underline{\text{iA}}} \qquad \phi^2 < 1 \quad \text{B}$$

$$\text{A} + \text{B} \Rightarrow |\phi| < 1$$

$$\gamma_k = Cov(\varepsilon_t, \varepsilon_{t-k})$$

$$\gamma_1 = Cov(\varepsilon_t, \varepsilon_{t-1})$$

$$Cov(\varepsilon_{t+1}, \varepsilon_t) = \gamma_1$$

$$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + u_t \longrightarrow$$

$$Cov(\varepsilon_t, \varepsilon_{t-k}) = Cov(\mu + \varphi\varepsilon_{t-1} + u_t, \varepsilon_{t-k})$$

$$\Rightarrow \gamma_k = \varphi\, Cov(\varepsilon_{t-1}, \varepsilon_{t-k}) + Cov(u_t, \varepsilon_{t-k})$$

$$\Rightarrow \gamma_k = \varphi \gamma_{k-1} + 0$$

$$\Rightarrow \quad \gamma_k = \phi \, \gamma_{k-1}$$

$$\gamma_1 = \phi \gamma_0 = \phi \frac{\sigma^2}{1-\phi^2}$$

$$\gamma_2 = \phi \gamma_1 = \phi^2 \gamma_0$$

$$\vdots$$

$$\gamma_k = \phi^k \gamma_0$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k \longleftarrow$$

(AUTOCORRELATION FUNCTION)

<u>ACF</u>

$\rho_k^{\bullet}$ = PARTIAL AUTOCOR. OF ORDER k

= AUTOCOR. του ε_t και ε_{t-k} αφού λάβω υπ' όψιν μου τα AUTOCOR. του $(\varepsilon_t, \varepsilon_{t-1}), (\varepsilon_t, \varepsilon_{t-2}), \ldots, (\varepsilon_t, \varepsilon_{t-k+1})$

$$\varepsilon_t = c^{*} + \rho_k \varepsilon_{t-k} + v_t$$

$$\varepsilon_t = c^{**} + \delta_1 \varepsilon_{t-1} + \delta_2 \varepsilon_{t-2} + \ldots + \delta_{k-1} \varepsilon_{t-k+1} + \rho_k^{\bullet} \varepsilon_{t-k} + \hat{v}_t$$

$$\rho_1^* = \rho_1 = \varphi$$

$$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + \rho_2^* \varepsilon_{t-2} + u_t$$

ε_t όταν ΕΧ(1) $\quad \varepsilon_t = \mu + \varphi \varepsilon_{t-1} + u_t \quad \Rightarrow$

$$\Rightarrow \rho_2^* = 0 = \rho_3^* = \rho_4^* = \dots$$

$$\begin{cases} \underset{1\times 1}{y_t} = \underset{1\times k}{x_t}\underset{k\times 1}{\beta} + \varepsilon_t \\ \varepsilon_t = \varphi\,\varepsilon_{t-1} + u_t, \quad u_t \overset{iid}{\sim} (0, \sigma^2) \end{cases}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon) = 0$$

$$V(\varepsilon) = E(\varepsilon\varepsilon')$$

$$= E\begin{pmatrix} \varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \cdots & \varepsilon_1\varepsilon_n \\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \varepsilon_2\varepsilon_3 & \cdots & \varepsilon_2\varepsilon_n \\ \vdots \\ \varepsilon_n\varepsilon_1 & \varepsilon_n\varepsilon_2 & \cdots & \cdots & \varepsilon_n^2 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \vdots \\ \gamma_{n-1} & \cdots & \cdots & & \gamma_0 \end{pmatrix} =$$

$$= \gamma_0 \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \rho_1 & & \rho_{n-2} \\ \vdots & & & & \\ \rho_{n-1} & \rho_{n-2} & \cdots & & 1 \end{pmatrix} =$$

$$= \frac{\sigma^2}{1-\phi^2} \begin{pmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{n-1} \\ \phi & 1 & \phi & \cdots & \phi^{n-2} \\ \vdots & & & & \\ \phi^{n-1} & \phi^{n-2} & \cdots & & 1 \end{pmatrix} = \sigma^{*2} \Omega$$

$V(\varepsilon) \neq \sigma^2 I_n \Rightarrow \hat{\beta} = (X'X)^{-1} X'y$ is UNBIASED BUT INEFFICIENT.

$$\varepsilon_t = \mu + u_t - \vartheta u_{t-1} \qquad MA(1)$$

$$E(\varepsilon_t) = \mu + 0 - \vartheta \cdot 0 = \mu \qquad \forall \vartheta$$

$$V(\varepsilon_t) = V(\mu + u_t - \vartheta u_{t-1}) = V(u_t) + \vartheta^2 V(u_{t-1})$$
$$- 2\vartheta\, Cov(u_t, u_{t-1}) = \sigma^2 + \sigma^2\vartheta^2 - 0 \Rightarrow$$

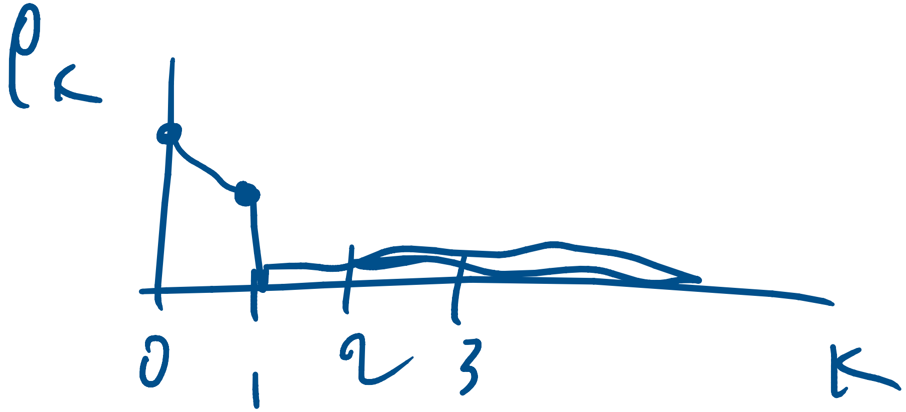
$$\Rightarrow V(\varepsilon_t) = \sigma^2(1+\vartheta^2) \qquad \forall \vartheta,$$

$\Rightarrow$ MA(1) ΠΑΝΤΑ ΣΤΑΣΙΜΟ ($\forall \vartheta$ is station 2nd order)

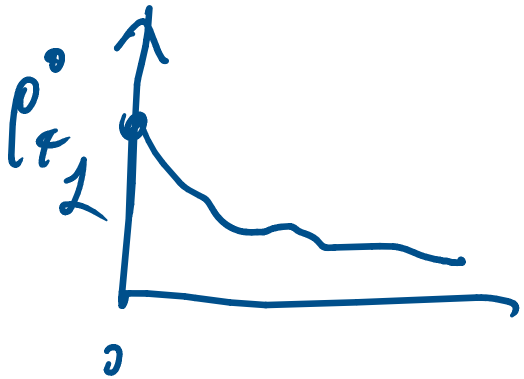
$$Cov(\varepsilon_t, \varepsilon_{t-k}) = \gamma_k = \begin{cases} -\dfrac{\vartheta}{(1+\vartheta^2)\sigma^2} & k=1 \\ 0 & \forall k>1 \end{cases}$$

ρ

$$\rho_k = \begin{cases} -\dfrac{\theta}{1+\theta^2} & k=1 \\ 0 & k \geq 2 \end{cases}$$

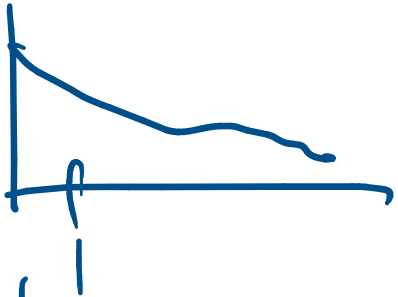


$$\rho_k^* = \theta^k \cdot ct.$$

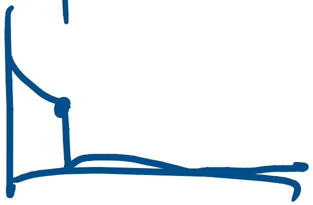


AR(1)

ρ_k

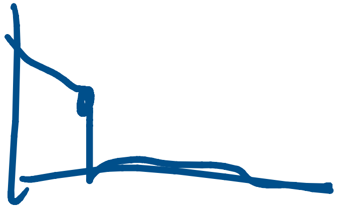


ρ^*_τ

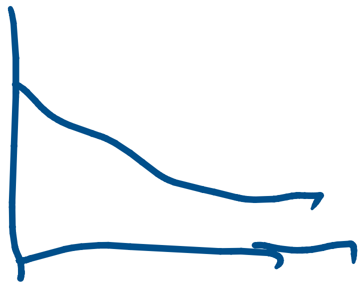


MA(1)

ρ_τ



ρ^*_τ

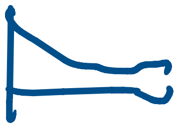


ARMA(1,1)

$$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + \eta_t - \vartheta \varepsilon_{t-1}$$

ρ_k

ρ^*_τ



AR(p), MA(q), ARMA(p, q)

ESTIMATION VIA MAX. LIKELIHOOD.

$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$ $E(\varepsilon) = 0$, $V(\varepsilon) = \sigma^2 I_n$

$\hat{\beta} = (X'X)^{-1}X'Y$ is the estimator

that minimizes $\sum_{t=1}^{n} \varepsilon_t^2 = \varepsilon'\varepsilon$

$\varepsilon \sim N(0, \sigma^2 I_n)$

$\Rightarrow L(y_1, y_2, \dots, y_n)$, $\max\limits_{\beta, \sigma^2} L(y_1, y_2, \dots, y_n) \longrightarrow \hat{\beta}_{MLE} = \hat{\beta}$

$\hat{\sigma}^2 = \frac{1}{n} \sum^{n} e^2$

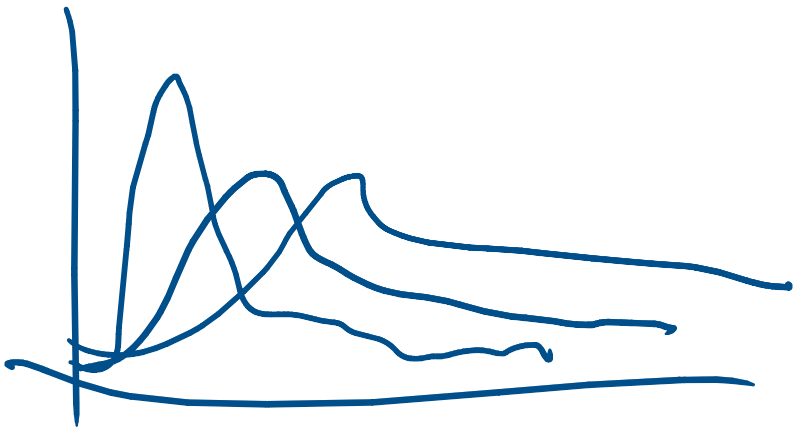
$X \sim \chi^2_n$

$Y \sim \chi^2_k$

X, Y INDEP.

$\Rightarrow \dfrac{X/n}{Y/k} \sim F(n, k)$

F



$$\rho_k = \operatorname{corr}(y_t, y_{t-k})$$

$$H_0: \rho_k = 0 \quad \text{v.s.} \quad H_1: \rho_k \neq 0$$

$$\sqrt{n}\,\hat{\rho}_k \overset{H_0}{\underset{A}{\sim}} N(0,1) \Rightarrow$$

$$\Rightarrow n\,\hat{\rho}_k^2 \overset{H_0}{\underset{A}{\sim}} \chi^2_1$$

$$\operatorname{cov}(\hat{\rho}_k, \hat{\rho}_\lambda) = 0 \quad \forall k \neq \lambda$$

TESTING AUTOCOR.

$H_0: \rho_k = 0$ v.s $H_1: \rho_k \neq 0$

$$\hat{\rho}_k \overset{A}{\underset{H_0}{\sim}} N\left(0, \frac{1}{n}\right) \Longleftrightarrow \sqrt{n}\,\hat{\rho}_k \overset{A}{\underset{H_0}{\sim}} N(0,1)$$

Reject H_0 iff

$$|\sqrt{n}\,\hat{\rho}_k| > 1.96 \quad (5\% \text{ size 2-sided})$$

$$\Rightarrow \quad n\hat{\rho}_k^2 \overset{A}{\underset{H_0}{\sim}} \chi^2_1 \Rightarrow \text{Reject } H_0 \text{ iff } n\hat{\rho}_k^2 > \chi^2_{1,5\%}$$

$$Q_k = n\left(\hat{\rho}_1^2 + \hat{\rho}_2^2 + \cdots + \hat{\rho}_k^2\right) \overset{H_0}{\underset{A}{\sim}} \chi^2_k$$

$H_0: \rho_1 = \rho_2 = \cdots = \rho_k = 0$ v.s. H_1: At least one $\neq 0$

$$y_t = x_t' \hat{b} + \hat{\varepsilon}_t$$

$$\hat{\varepsilon}_t^2 = x_t'\delta + \alpha_1 \hat{\varepsilon}_{t-1} + \alpha_2 \hat{\varepsilon}_{t-2} + \cdots + \alpha_k \hat{\varepsilon}_{t-k}$$

$\downarrow R^2$ $\quad nR^2 \overset{H_0}{\underset{A}{\sim}} \chi^2_k$ $\quad H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$

$$y_t = x_t' \beta + \varepsilon_t$$

$$\varepsilon_t = u_t - \vartheta u_{t-1} \qquad u_t \overset{iid}{\sim} (0, \sigma^2)$$

MOVING AVERAGE of order 1 $\equiv$ MA(1)

ΤΟ ΕΧΟΥΜΕ ΠΕΙ,

PORTMANTEAU $\quad Q_K$

BRUSH-GODFREI $\quad$ LM