Let $R_t = \dfrac{P_t - P_{t-1}}{P_{t-1}} \approx \ln P_t - \ln P_{t-1}$

P_t = Price of asset (RISKY) at time t

P_{t-1} = " " " " " " " " $t-1$

R_t = Return of asset sold at time t
bought at time $t-1$

Now the EXCESS Return of the asset r_t is given by

$$r_t = R_t - R^F_{t-1} \quad \text{where}$$

R^F_{t-1} is the return of the RISK FREE asset at time $t-1$.

NOTICE:

Interest rates and bond returns are quoted in an ANNUAL BASE e.g. 3% 6-month Bond means 3% in an annual Base.

Hence the risky asset Return and the safe asset Return must be at the same frequency. To transform the annual return in the frequency we need we have to:

Example Let the interest rate (monthly) is 10% (annualised). To transform it in a monthly base we have to $r = \sqrt[12]{1+0.1} - 1$ $= 0.00797 = 0.797\%$

$$y_t = \underset{\substack{\wedge \\ x_{t1}=1}}{\beta_1} + x_{t2}\beta_2 + \cdots + x_{tk}\beta_k + \boxed{\varepsilon_t}$$

$$E(\varepsilon_t) = 0, \quad V(\varepsilon_t) = \sigma^2$$

$$\forall\, t = 1, 2, \ldots, \underline{\underline{n}}$$

$$\left.\begin{array}{l}
y_1 = b_1 + x_{12} b_2 + \dots + x_{1k} b_k + \varepsilon_1 \\
y_2 = b_1 + x_{22} b_2 + \dots + x_{2k} b_k + \varepsilon_2 \\
y_3 = b_1 + x_{32} b_2 + \dots + x_{3k} b_k + \varepsilon_3 \\
\vdots \\
y_n = b_1 + x_{n2} b_2 + \dots + x_{nk} b_k + \varepsilon_n
\end{array}\right| \Rightarrow$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} b_1 + \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ \vdots \\ x_{n2} \end{pmatrix} b_2 + \dots + \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{pmatrix} b_k + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \dots \varepsilon_n \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{12} & \cdots & x_{1k} \\ 1 & x_{22} & \cdots & x_{2k} \\ 1 & x_{32} & \cdots & x_{3k} \\ \vdots & & \vdots & \vdots \\ 1 & x_{n2} & \cdots & x_{nk} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\Rightarrow \underset{n\times 1}{Y} = \underset{n\times k}{X}\ \underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

$$E(\varepsilon) = E\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix} = \begin{pmatrix}E(\varepsilon_1)\\ E(\varepsilon_2)\\ \vdots\\ E(\varepsilon_n)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

$$V(\varepsilon) = E(\varepsilon\varepsilon') =$$

$$= E\left[\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\right] =$$

$$= E\begin{pmatrix}\varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \cdots & \varepsilon_1\varepsilon_n\\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \varepsilon_2\varepsilon_3 & \cdots & \varepsilon_2\varepsilon_n\\ \varepsilon_3\varepsilon_1 & \varepsilon_3\varepsilon_2 & \varepsilon_3^2 & \cdots & \varepsilon_3\varepsilon_n\\ \vdots & & & & \\ \varepsilon_n\varepsilon_1 & \varepsilon_n\varepsilon_2 & \cdots & \cdots & \varepsilon_n^2\end{pmatrix} = \begin{pmatrix}\sigma^2 & 0 & 0 & \cdots & 0\\ 0 & \sigma^2 & 0 & \cdots & 0\\ 0 & & \ddots & & \vdots\\ \vdots & & & & \\ 0 & \cdots & \cdots & 0 & \sigma^2\end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & 0 & & \ddots & 1 \end{pmatrix} = \sigma^2 I_n$$

$$V(\varepsilon_t) = \sigma^2 \Rightarrow V(\varepsilon_t) = E(\varepsilon_t - \underbrace{E(\varepsilon_t)}_{0})^2 =$$

$$= E(\varepsilon_t^2) = \sigma^2 \quad \forall t = 1, 2, \ldots, n$$

X ΓΕΝΙΚΑ

ΤΥΧΑΙΑ ΜΕΤΑΒΛΗΤΗ

$$V(X) = E(X - E(X))^2$$

Αν $E(X) = 0$

$$\Rightarrow V(X) = E(X^2)$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\overset{?}{b}} + \underset{n\times 1}{\overset{?}{\varepsilon}}, \quad E(\varepsilon)=0, \quad V(\varepsilon)=\sigma^2 I_n$$

$\boxed{k+n} \leftarrow$ ΑΓΝΩΣΤΟΥΣ

$$\min_b \sum_{t=1}^{n} \varepsilon_t^2 = \min_b \sum_t \left(y_t - b_1 - x_{t2}b_2 - \dots - x_{tk}b_k\right)^2$$

$$= \min_b (\varepsilon_1\ \varepsilon_2 \dots \varepsilon_n)\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix} = \min_b \underset{1\times n}{\varepsilon'}\,\underset{n\times 1}{\varepsilon} =$$

$$= \min_b (Y - Xb)'(Y - Xb)$$

$$S = (y - X b)'(y - X b)$$

$$\frac{\partial S_{1\times 1}}{\partial b_{k\times 1}} = 0 \Rightarrow \cancel{2}(X'X)\hat{b} - \cancel{2}X'y = 0$$

$$\Rightarrow (X'X)\hat{b} - X'y = 0 \Rightarrow \underbrace{\underbrace{(X'X)}_{k\times n \; n\times k}}_{k\times k}\underbrace{\hat{b}}_{k\times 1} = \underbrace{\underbrace{X'y}_{k\times n \; n\times 1}}_{k\times 1}$$

$$\Rightarrow \boxed{\hat{b} = (X'X)^{-1}X'y}$$

NORMAL EQUAT.

IFF $(X'X)^{-1}$ ΥΠΑΡΧΕΙ $\Leftrightarrow$

$\Longleftrightarrow \quad |\underbrace{X'_{k\times n}\, X_{n\times k}}_{k\times k}| \neq 0$ δηλαδή $X'X$ ΜΗ-ΙΔΙΑΖΟΥΣΑ

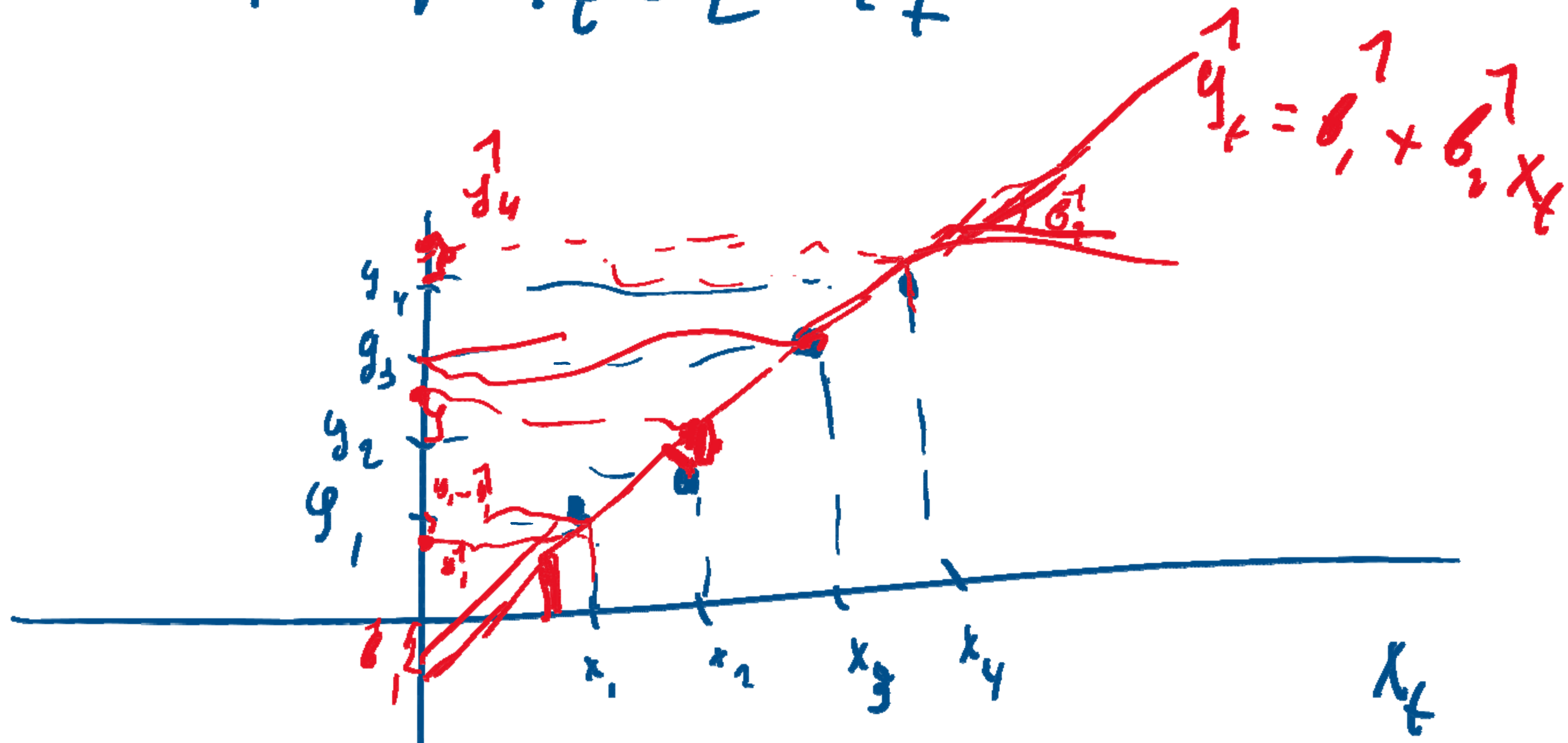
$\Longleftrightarrow \quad r(\underset{n\times k}{X}) = \qquad \Longleftrightarrow$

$\uparrow$

RANK OF X

ΟΙ Κ ΕΞΩΓΕΝΕΙΣ ΜΕΤΑΒΛ. ΠΡΕΠΕΙ ΝΑ ΓΡΑΜΜΙΚΑ ΑΝΕΞΑΡΤΗΤΕΣ. ΔΗΛΑΔΗ ΝΑ ΜΗΝ ΕΧΩ ΠΟΛΛΥΣΓΡΑΜΜΙΚΟΤΗΤΑ (MULTICOLINEARITY)

$$y_t = \beta_1 + x_t \beta_2 + \varepsilon_t$$



$$\hat{\varepsilon}_t = y_t - \hat{y}_t = y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t$$

$$\hat{b} = (X'X)^{-1} X'y$$

$$\hat{y} = X\hat{b}$$

$\hat{y}$ FITTED VALUES

$$\begin{cases} \hat{y}_1 = \hat{b}_1 + x_{12}\hat{b}_2 + \dots + x_{1k}\hat{b}_k \\ \hat{y}_2 = \hat{b}_1 + x_{22}\hat{b}_2 + \dots + x_{2k}\hat{b}_k \\ \vdots \\ \hat{y}_n = \hat{b}_1 + x_{n2}\hat{b}_2 + \dots + x_{nk}\hat{b}_k \end{cases}$$

$$\hat{\varepsilon} = \underset{=}{y} - \hat{y}$$

↑ ΚΑΤΑΛΟΙΠΑ (RESIDUALS)

$$\underset{n\times 1}{\hat{y}} = X\hat{b} = \underbrace{X(X'X)^{-1}X'}_{n\times n}\underset{n\times 1}{y} = P_x y$$

$$P_x = X(X'X)^{-1}X'$$ ΣΥΜΜΕΤΡΙΚΗ + ΤΑΥΤΟΔΥΝΑΜΗ

P_x ΣΥΜΜΕΤΡΙΚΗ $\equiv P_x' = P_x$

$$P_X' = \left[X(X'X)^{-1}X'\right]' = \qquad ((AB\Gamma)' = \Gamma'B'A')$$

$$= (X')'\left(\left((X'X)^{-1}\right)'\right)'X'$$

$$= X\left(\left((X'X)'\right)^{-1}\right)'X' =$$

$$= X\left(X'(X')'\right)^{-1}X' = X(X'X)^{-1}X' = P_X$$

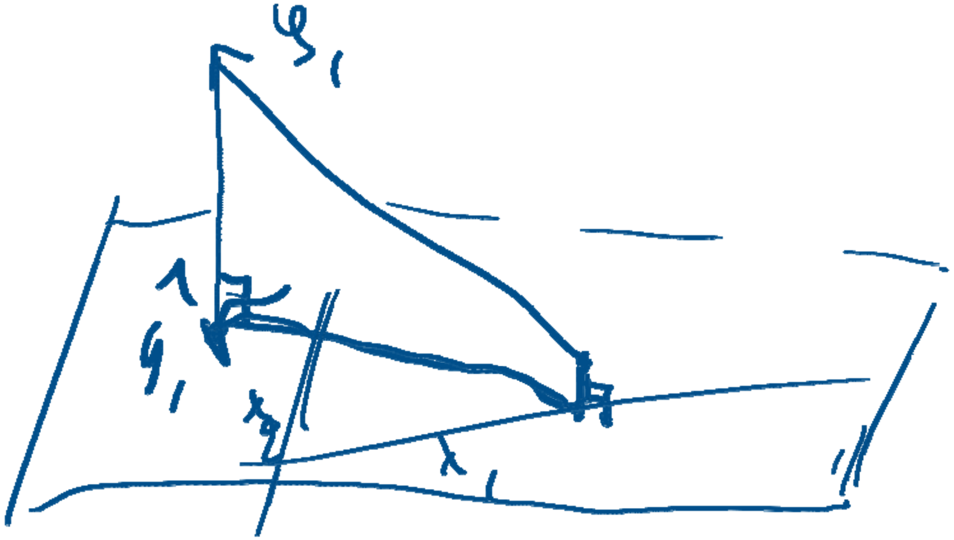
ΟΝΤΩΣ P_X ΣΥΜΜΕΤΡΙΚΗ

P_X ΤΑΥΤΟΔΥΝΑΜΗ $\Leftrightarrow P_X^2 = P_X \cdot P_X = P_X$

$$P_X^2 = P_X \cdot P_X = \left(X(X'X)^{-1}X'\right)\left(X(X'X)^{-1}X'\right)$$

$$= X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{=I_k}X' =$$

$$= X(X'X)^{-1}I_k X' = X(X'X)^{-1}X' = P_X$$



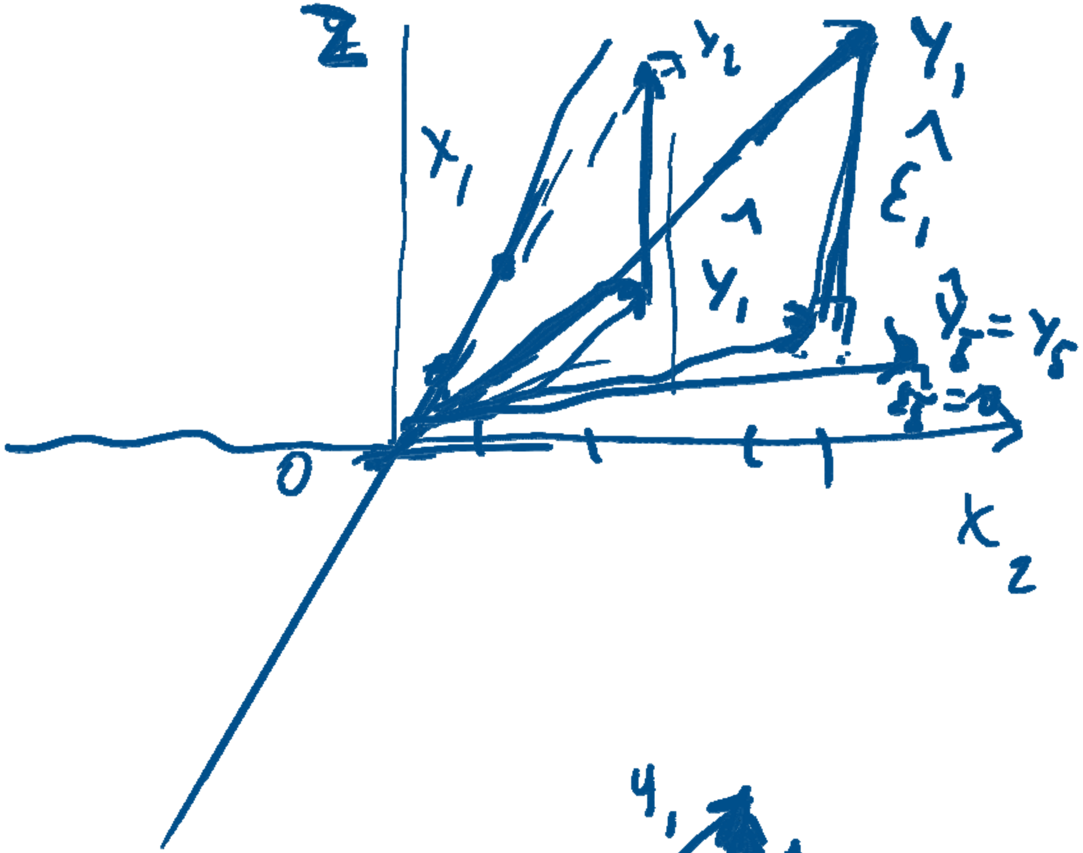
$$\hat{y} = P_X \, \underline{\underline{y}}$$

P_X SYMMETRY

$$r(P_X) = r\left(X(X'X)^{-1}X'\right) =$$

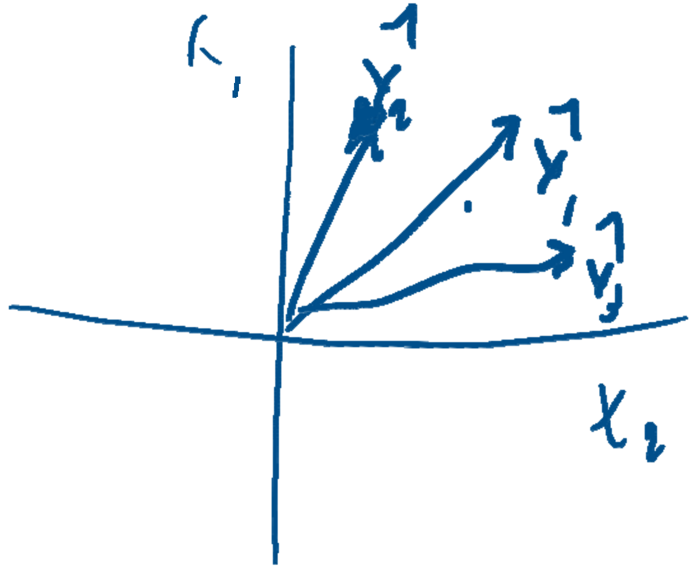
$$= tr\left(X(X'X)^{-1}X'\right) = tr\left((X'X)^{-1}X'X\right) =$$

$$= tr(I_K) = K$$



$$\hat{\varepsilon}_1 = y_1 - \hat{y}_1$$

$$\hat{\varepsilon}_1 + \hat{y}_1 = y_1$$



ΓΕΝΙΚΑ

$\underset{n \times k}{r(A)}$ = RANK (A) ΒΑΘΜΟΣ ΤΗΣ Α

= ΑΡΙΘΜΟΣ ΤΩΝ ΓΡΑΜΜΙΚΑ ΑΝΕΞΑΡΤΗΤΩΝ ΓΡΑΜΜΩΝ Ή ΣΤΗΛΩΝ

$r(A) \leq \min(n, k)$

Π.χ. $\underset{10 \times 5}{A}$ $r(A)$ ΤΟ ΠΟΛΥ ΘΑ ΕΙΝΑΙ 5

$A_{n \times n}$ (ΤΕΤΡΑΓΩΝΙΚΗ)

ΤΟΤΕ $tr(A) = trace(A) =$

$= \text{ΙΧΝΟΣ}(A) = \sum_{i=1}^{n} \alpha_{ii} = \alpha_{11} + \alpha_{22} + \ldots + \alpha_{nn}$

= ΑΘΡΟΙΣΜΑ ΤΩΝ ΣΤΟΙΧΕΙΩΝ ΤΗΣ ΚΥΡΙΑΣ ΔΙΑΓΩΝΙΟΥ.

$\underset{n\times n}{A}$ ΣΥΜΜΕΤΡΙΚΗ $\underline{\underline{\text{ΑΝ}}}$ $A' = A$

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$$

$$tr(AB\Gamma) = tr(B\Gamma A) = tr(\Gamma AB)$$

$$r(AA') = r(A'A) = r(A)$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon) = \underline{0} \qquad V(\varepsilon) = \sigma^2 I_n$$

$$\hat{\beta} = (X'X)^{-1}X'Y = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\varepsilon$$

$$= \beta + (X'X)^{-1}X'\varepsilon$$

$$E(\hat{\beta}) = E\left[\beta + (X'X)^{-1}X'\varepsilon\right] =$$

$$= E(\beta) + E\left[(X'X)^{-1}X'\varepsilon\right] =$$

$$= \beta + (X'X)^{-1}X'\underset{\underset{0}{\shortparallel}}{E(\varepsilon)} = \beta$$

$\hat{\beta}$ UNBIASED ESTIMATOR OF β.

$$V(\hat{\beta}) = E\left\{ \left[\hat{\beta} - E(\hat{\beta})\right] \left[\hat{\beta} - E(\hat{\beta})\right]' \right\}$$

$$= E\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)' \right]$$

$$\hat{\beta} - \beta = (X'X)^{-1} X' \varepsilon$$

$$\Rightarrow$$

$$V(\hat{\beta}) = E\left[(X'X)^{-1} X'\varepsilon \left[(X'X)^{-1} X'\varepsilon\right]' \right]$$

$$= E\left[(X'X)^{-1} X'\varepsilon\varepsilon' X (X'X)^{-1} \right] =$$

$$= (X'X)^{-1} X' E(\varepsilon\varepsilon') X (X'X)^{-1} =$$

$$= (X'X)^{-1} X' \sigma^2 I_n X (X'X)^{-1} =$$

$$= \sigma^2 (X'X)^{-1} \cancel{X'X (X'X)^{-1}} =$$

$$= \sigma^2 (X'X)^{-1}$$

$$E(\hat{\beta}) = \beta, \quad V(\hat{\beta}) = \sigma^2 (X'X)^{-1}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{b} + \underset{n\times 1}{\varepsilon}$$

1) Η ΣΧΕΣΗ ΣΩΣΤΗ

2) $r(X) = k$ (NO MULTICOL.)

3) $E(\varepsilon) = 0$, $V(\varepsilon) = \sigma^2 I_n$

$$V(\varepsilon) = E(\varepsilon\varepsilon') = E\left[\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\right]$$

$$= E\begin{pmatrix} \varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \varepsilon_1\varepsilon_4 & \cdots & \varepsilon_1\varepsilon_n \\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \varepsilon_2\varepsilon_3 & \varepsilon_2\varepsilon_4 & \cdots & \varepsilon_2\varepsilon_n \\ \varepsilon_3\varepsilon_1 & \varepsilon_3\varepsilon_2 & \varepsilon_3^2 & \varepsilon_3\varepsilon_4 & \cdots & \varepsilon_3\varepsilon_n \\ \vdots & & & & \ddots & \\ \varepsilon_n\varepsilon_1 & \varepsilon_n\varepsilon_2 & \cdots & \cdots & \cdots & \varepsilon_n^2 \end{pmatrix} =$$

$$
= \begin{pmatrix} E\xi_1^2 & E\xi_1\xi_2 & \cdots & E\xi_1\xi_n \\ E\xi_2\xi_1 & E\xi_2^2 & \cdots & E\xi_2\xi_n \\ \vdots & & \ddots & \\ E\xi_n\xi_1 & \cdots & \cdots & E(\xi_n^2) \end{pmatrix} =
$$

$$
= \begin{pmatrix} V(\xi_1) & Cov(\xi_1,\xi_2) & \cdots & Cov(\xi_1,\xi_n) \\ Cov(\xi_2,\xi_1) & V(\xi_2) & \cdots & \\ \vdots & & \ddots & \\ Cov(\xi_n,\xi_1) & \cdots & \cdots & V(\xi_n) \end{pmatrix} =
$$

$$= \begin{pmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \sigma^2 \end{pmatrix} = \sigma^2 I_n$$

Theorem GAUSS-MARKOV

$\hat{\beta}$ BEST LINEAR UNBIASED ESTIMATOR of β.

$\hat{\beta} = (X'X)^{-1} X'Y$ (LINEAR IN Y)

BEST $\equiv$ $V(\hat{\beta}) \leq V(\tilde{\beta})$

ΟΠΟΥ $\tilde{\beta}$ ΟΙΠΟΙΟΣΔΗΠΟΤΕ ΓΡΑΜΜΙΚΟΣ ΑΜΕΡΟΛΗΠΤΟΣ ΕΚΤΙΜΗΤΗΣ ΤΟΥ β.

ΕΣΤΩ $\tilde{\beta}$ ΑΛΛΟΣ LINEAR + UNBIASED ESTIMATOR $\Rightarrow$

$$V(\hat{\beta}) \leq V(\tilde{\beta}) \Rightarrow$$

$V(\hat{\beta}) - V(\tilde{\beta})$ NEGATIVE DEFINITE MATRIX.

$$V(\hat{\beta}) = \underbrace{\sigma^2 (\underset{k\times n\; n\times k}{X'X})^{-1}}_{k\times k}$$

$$V(\hat{\beta}) = E\left[\underbrace{\underset{k\times 1}{(\hat{\beta}-\beta)}\,\underset{1\times k}{(\hat{\beta}-\beta)'}}_{k\times k}\right]$$

$$= \begin{pmatrix} V(\hat{\beta}_1) & Cov(\hat{\beta}_1,\hat{\beta}_2) & \cdots & Cov(\hat{\beta}_1,\hat{\beta}_k) \\ Cov(\hat{\beta}_2,\hat{\beta}_1) & V(\hat{\beta}_2) & \cdots & Cov(\hat{\beta}_2,\hat{\beta}_k) \\ \vdots & & & \\ Cov(\hat{\beta}_k,\hat{\beta}_1) & \cdots & \cdots & V(\hat{\beta}_k) \end{pmatrix}$$

$$\sum_{t=1}^{n} \hat{\varepsilon}_t^2 = \hat{\varepsilon}'\hat{\varepsilon}$$

$$\hat{\varepsilon} = Y - X\hat{b} = Y - X(X'X)^{-1}X'Y =$$

$$= \underbrace{\left[I_n - X(X'X)^{-1}X' \right]}_{M} Y = MY$$

$$M' = \left(I_n - X(X'X)^{-1}X'\right)' = I_n' - \left(X(X'X)^{-1}X'\right)' =$$

$$= I_n - (X')'\left((X'X)^{-1}\right)'X' = I_n - X(X'X)^{-1}X' = M$$

M ΣΥΜΜΕΤΡΙΚΗ

$$M \cdot M = [I_n - X(X'X)^{-1}X'][I_n - X(X'X)^{-1}X']$$

$$= I_n - X(X'X)^{-1}X' - X(X'X)^{-1}X'$$

$$+ X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}X' =$$

$$= I_n - X(X'X)^{-1}X' - \cancel{X(X'X)^{-1}X'} + \cancel{X(X'X)^{-1}X'}$$

$$= I_n - X(X'X)^{-1}X' = M \Rightarrow$$

M ΤΑΥΤΟΔΥΝΑΜΗ.

$$M'X = MX = (I_n - X(X'X)^{-1}X')X =$$

$$= X - X\underbrace{(X'X)^{-1}X'X}_{I_k} = X - X = 0$$

M ΟΡΘΟΓΩΝΙΑ ΤΗΣ X.

$$\hat{\varepsilon} = My \Rightarrow \hat{\varepsilon} = M(X\beta + \varepsilon) \Rightarrow$$

$$\Rightarrow \hat{\varepsilon} = \underbrace{MX}_{=0}\beta + M\varepsilon \Rightarrow \boxed{\hat{\varepsilon} = M\varepsilon}$$

$$M = I_n - X(X'X)^{-1}X'$$

$$r(M) \underset{\text{ταυτ.}}{\overset{\text{sym.}}{=\!=}} tr(M) = tr\left(I_n - X(X'X)^{-1}X'\right)$$

$$= tr(I_n) - tr\left(X(X'X)^{-1}X'\right) =$$

$$= n - tr\left((X'X)^{-1}X'X\right) = n - tr(I_k) =$$

$$= n - k < n \Rightarrow |M| = 0 \Rightarrow M^{-1} \text{ ΔΕΝ ΥΠΑΡΧΕΙ}$$