PROBLEM 1

Let $U \xrightarrow{S} V \xrightarrow{T} W$ be linear maps between finite-dimensional vector spaces over $\mathbb{R}$. Show that

a) $rank(T \circ S) \leq rank(S)$

b) $rank(T \circ S) \leq rank(T)$

c) give an example of linear maps that satisfy $rank(T \circ S) < rank(S)$

d) give an example of linear maps that satisfy $rank(T \circ S) = rank(S)$

PROBLEM 2

Let $V \xrightarrow{T} V$ be a linear map between finite-dimensional vector spaces over $\mathbb{R}$,with rank(T)=1. Show that

a) there exists a number $\alpha \in \mathbb{R}$ such that $T^2 = \alpha T$

b) if $\alpha \neq 1$ then T is invertible.

PROBLEM 3

Let $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ be the Euclidean norm function, defined by $f(x) = |x|$.

a) Show that f is a convex function

b) show that f is strictly quasi-convex, namely that

$$f(tx + (1-t)y) < \max\{f(x), f(y)\} \text{ for all } 0 < t < 1, x \neq y$$

PROBLEM 4

Let $H = \{x \in \mathbb{R}^n : px = \theta\}$ be the hyperplane defined by $p \in \mathbb{R}^n, p \neq 0, \theta \in \mathbb{R}$.

Let $a \in \mathbb{R}^n$ be any point not in H,and let $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ be the function defined by $f(x) = |x - a| =$ Euclidean distance of x from a.

Consider the minimization problem (f, H),with objective function f and feasible set H .

a) show that (f, H) has a global minimum b

b) show that the global minimum b is unique

c)give a formula for the minimal distance $f(b)$ that involves only p,a,θ

PROBLEM 5

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be a linear map. Show that there exists a constant $C \geq 0$ such that $|T(x)| \leq C|x|$ for all $x \in \mathbb{R}^n$.

PROBLEM 6

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be a linear map. Consider the maximization problem (f,S),with objective function $f(x) = \frac{|T(x)|}{|x|}$,and feasible set $S = \{x \in \mathbb{R}^n : x \neq 0\}$.

a) show that (f,S) has a global maximum

Denote by $\nu(T)$ the maximum value of $\frac{|T(x)|}{|x|}$ on S.

b) Show that ν behaves like a norm, i.e.

b1) $\nu(T) \geq 0$

b2) $\nu(T) = 0$ iff $T = 0$

b3) $\nu(\lambda T) = |\lambda| \nu(T), \forall \lambda \in \mathbb{R}$

b4) $\nu(T + T') \leq \nu(T) + \nu(T')$

for any two linear maps $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m, \mathbb{R}^n \xrightarrow{T'} \mathbb{R}^m$

b5) if $m = n$ then $\nu(S \circ T) \leq \nu(S)\nu(T)$

for any two linear maps $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n, \mathbb{R}^n \xrightarrow{S} \mathbb{R}^n$

PROBLEM 7

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $f(x) = \alpha x_1 + \log(x_2 x_3 ... x_n)$

constraints $p_1 x_1 + ... + p_n x_n \leq M, x_1 \geq 0, x_2 \geq 0, .., x_n \geq 0$

variables $x_1, x_2, .., x_n$

parameters $\alpha, p_1, ..., p_n, M$

conditions on parameters: $\alpha > 0, p_1 > 0, p_2 > 0, ..., p_{n-1} > 0, M > 0$.

p_n is a free parameter. It can be positive, zero, or negative.

PROBLEM 8

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $p_1(-4t_1 - \frac{3}{2}t_2 - 1 - x_1) + p_2(2t_1 + t_2 + 1 - x_2))$

constraints $t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1, x_1 \geq 0, x_2 \geq 0$

variables x_1, x_2, t_1, t_2

parameters p_1, p_2

conditions on parameters: $p_1 > 0, p_2 > 0$.

PROBLEM 9

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $p_1 y_1 + p_2 y_2$

constraints $y_1 + y_2 \leq 0, y_1 + 3y_2 \leq 4, 2y_1 + 3y_2 \leq 1$

variables y_1, y_2

parameters p_1, p_2

conditions on parameters: $p_1 > 0, p_2 > 0$.

PROBLEM 10

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $\alpha\left(\log A_1 + \log X_1\right) + \left(1-\alpha\right) X_2$

Constraints

$$A \leq \sqrt{2X},\ A_1 + A_2 \leq A,\ X_1 + X_2 + X \leq 4$$
$$A \geq 0, A_1 \geq 0,\ A_2 \geq 0,\ X_1 \geq 0,\ X_2 \geq 0,\ X \geq 0$$

variables A_1, A_2, A, X_1, X_2, X

parameters α

conditions on parameters: $0 < \alpha < 1$.