**please answer all questions.**

## PROBLEM 0

Consider two maximization problems $(S,f),(S,g)$,where $S$ is any set and $S\xrightarrow{f}\mathbb{R}, S\xrightarrow{g}\mathbb{R}$ are real-valued functions. Suppose that $f,g$ create the same order on the set $S$ (they represent the same preferences).Are the two problems equivalent, i.e. do they have the same set of global maxima? Proof or counterexample.

**In some of the following problems, it is convenient to use your answer to problem 0**

## PROBLEM 1

**find all global maxima of the following maximization problem or show that none exist.**

| maximization problem | |
|---|---|
| Objective function | $\mathbb{R}^n_+\xrightarrow{f}\mathbb{R}, f(A)=A_1+\sum_{i=2}^{n}\alpha_i\log A_i$ |
| Constraints | $\sum_{i=1}^{n}p_iA_i\le M, A_i\ge 0$ for all $i$ |
| Variables | $A_1,A_2,\dots,A_n$ |
| Parameters | $p_1,p_2,\dots,p_n,M,\alpha_2,\dots,\alpha_n$ |
| Conditions on parameters:all strictly positive | |

(1)

answer to problem 1

let $\alpha=\alpha_2+\alpha_3+\dots+\alpha_n$

$$A=\begin{cases}[0,\dfrac{\alpha_2M}{\alpha p_2},\dfrac{\alpha_3M}{\alpha p_3},\dots,\dfrac{\alpha_nM}{\alpha p_n}] & \text{if}\quad M\le\alpha p_1\\[2ex] [\dfrac{M-\alpha p_1}{p_1},\dfrac{\alpha_2p_1}{p_2},\dfrac{\alpha_3p_1}{p_3},\dots,\dfrac{\alpha_np_1}{p_n}] & \text{if}\quad M\ge\alpha p_1\end{cases}$$

## PROBLEM 2

**find all global maxima of the following maximization problem, or show that none exist**

maximization problem

Objective function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}, f(B) = B_1 + (B_2)^\beta$

Constraints $A_1 \geq \theta, A_1 + B_1 \leq 5, B_2 \leq 1, A_1 \geq 0, B_1 \geq 0, B_2 \geq 0$

Variables $A_1, B_1, B_2$

Parameters $\theta, \beta$

conditions on parameters : $\beta > 0$

(2)

answer to problem 2

$\theta > 5$ no solution

for $\theta \leq 5, B_2 = 1$ and

$$[A_1, B_1] = \begin{cases} [\theta, 5-\theta] & if \quad \theta \geq 0 \\ [0,5] & if \quad \theta \leq 0 \end{cases}$$

## PROBLEM 3

### find all global maxima of the following minimization problem, or show that none exist

minimization problem

objective function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}, f(x) = w_1 x_1 + w_2 x_2$

constraints $(x_1)^\alpha + x_2 \geq \theta, x_1 \geq 0, x_2 \geq 0$

variables $x_1, x_2$

parameters $\alpha, \theta, w_1, w_2$

conditions on parameters : all strictly positive

(3)

answer to problem 3,case $\alpha < 1$

$$x = \begin{cases} [\left(\frac{\alpha w_2}{w_1}\right)^{\frac{1}{1-\alpha}}, \theta - \left(\frac{\alpha w_2}{w_1}\right)^{\frac{\alpha}{1-\alpha}}] & \text{if} \quad \left(\frac{\alpha w_2}{w_1}\right)^{\frac{\alpha}{1-\alpha}} \leq \theta \\ [\theta^{\frac{1}{\alpha}}, 0] & \text{if} \quad \left(\frac{\alpha w_2}{w_1}\right)^{\frac{\alpha}{1-\alpha}} \geq \theta \end{cases}$$

answer to problem 3,case $\alpha = 1$

$$x = \begin{cases} [\theta, 0] & \text{if} \quad w_1 < w_2 \\ \{[x_1, \theta - x_1], 0 \le x_1 \le \theta\} & \text{if} \quad w_1 = w_2 \\ [0, \theta] & \text{if} \quad w_1 > w_2 \end{cases}$$

answer to problem 3,case $\alpha > 1$

$$x = \begin{cases} [0, \theta] & \text{if} \quad \left(\frac{w_1}{w_2}\right)^{\frac{\alpha}{1-\alpha}} > \theta \\ \left\{[0, \theta], [\theta^{\frac{1}{\alpha}}, 0]\right\} & \text{if} \quad \left(\frac{w_1}{w_2}\right)^{\frac{\alpha}{1-\alpha}} = \theta \\ [\theta^{\frac{1}{\alpha}}, 0] & \text{if} \quad \left(\frac{w_1}{w_2}\right)^{\frac{\alpha}{1-\alpha}} < \theta \end{cases}$$

# PROBLEM 4

## find all global maxima of the following maximization problem, or show that none exist

maximization problem

Objective function $\mathbb{R}_+^2 \xrightarrow{f} \mathbb{R}, f(A_2, B_2) = \left(\min((A_2)^\alpha, (B_2)^\alpha)\right)^\beta$

Constraints

$$\begin{array}{l} \ln((B_1)^\gamma + 1) \ge s \\ A_1 + A_2 \le \theta\sqrt{2C} \\ B_1 + B_2 \le 1 \\ C \le 1/2 \\ A_1 \ge 0, A_2 \ge 0, B_1 \ge 0, B_2 \ge 0, C \ge 0 \end{array} \qquad (4)$$

Variables $A_1, A_2, B_1, B_2, C$

Parameters $s, \theta, \alpha, \beta, \gamma$

conditions on parameters:All strictly positive answer to problem 4

in all cases, $C = 1/2, A_1 = 0, A_2 = B_2$

Let $\psi(s) = (e^s - 1)^{1/\gamma}$

$$[B_1, B_2] = \begin{cases} [\psi(s), 1-\psi(s)] & \text{if} \quad s < \log 2, \theta \geq 1-\psi(s) \\ [\psi(s), \theta] & \text{if} \quad s < \log 2, \theta \leq 1-\psi(s) \\ [1, 0] & \text{if} \quad s = \log 2 \\ \text{NONE} & \text{if} \quad s > \log 2 \end{cases}$$

# PROBLEM 5

## find all global maxima of the following maximization problem, or show that none exist

maximization problem

Objective function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$f(K, L) = p \min\{K + \alpha L, \beta K + L\} - wL - rK$

Constraints $K \geq 0, L \geq 0$ (5)

Variables $K, L$

Parameters $p, w, r, \alpha, \beta$

conditions on parameters: all strictly positive

## ANSWER TO PROBLEM 5

Split the problem into two: cost minimization first and then profit maximization.

Cost minimization problem

Objective function $\mathbb{R}^2 \xrightarrow{\Psi} \mathbb{R}$

$\Psi(K, L) = wL + rK$

Constraints $K \geq 0, L \geq 0, \min\{K + \alpha L, \beta K + L\} \geq Y$

Variables $K, L$

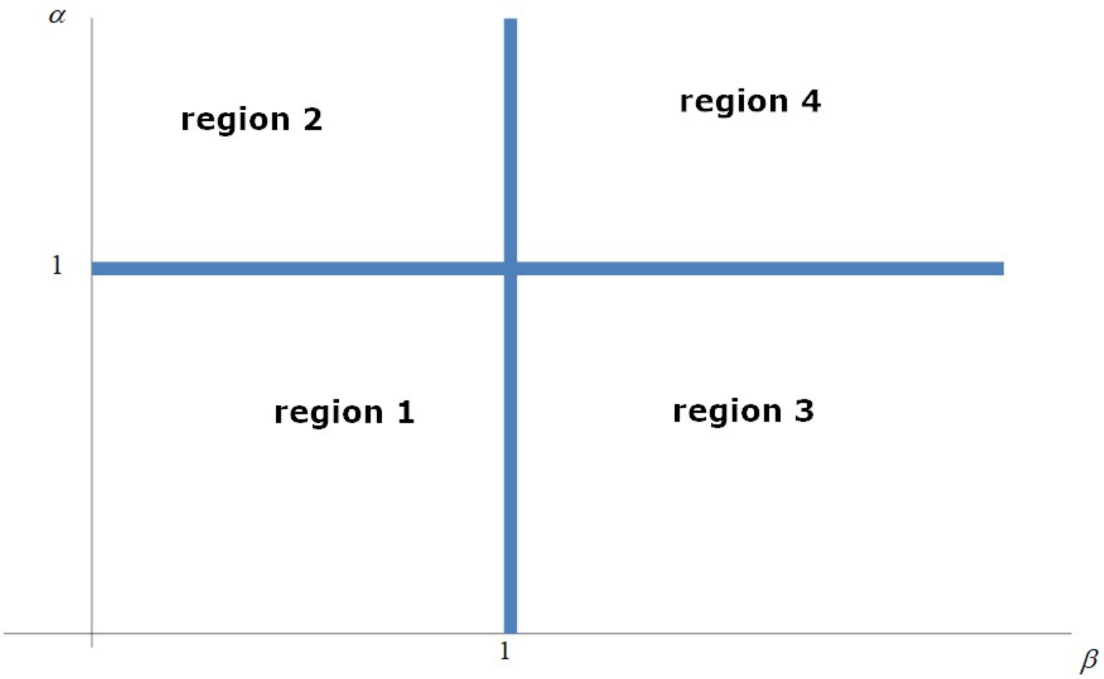
Parameters $w, r, \alpha, \beta, Y$

conditions on parameters: $w, r, \alpha, \beta$ all strictly positive, $Y \geq 0$

Let $C(w, r, \alpha, \beta, Y)$ be the value of the objective function $\Psi(K, L) = wL + rK$ at a solution $(K, L)$ of the cost minimization problem.

| Profit maximization problem |
|---|
| Objective function $\mathbb{R}_+ \xrightarrow{\Pi} \mathbb{R}$ |
| $\Pi(Y) = pY - C(w,r,\alpha,\beta,Y)$ |
| Constraints $Y \geq 0$ |
| Variables $Y$ |
| Parameters $p,w,r,\alpha,\beta$ |
| conditions on parameters:all strictly positive |

The solutions depend on the values of the parameters $\alpha,\beta$




Let $t = \frac{w}{r}, AC = \frac{C}{Y}$ .Then answer to problem 5,case $\alpha<1,\beta<1$

cost minimizing choices

$$[K,L]=\begin{cases}[0,Y/\alpha] & \text{if} \quad t<\alpha \\ [\frac{Y(\alpha-1)}{\alpha\beta-1},\frac{Y(\beta-1)}{\alpha\beta-1}] & \text{if} \quad \alpha<t<1/\beta \\ [Y/\beta,0] & \text{if} \quad t>1/\beta\end{cases}$$

$$AC=\begin{cases}w/\alpha & \text{if} \quad t<\alpha \\ r\frac{(\alpha-1)}{\alpha\beta-1}+w\frac{(\beta-1)}{\alpha\beta-1} & \text{if} \quad \alpha<t<1/\beta \\ r/\beta & \text{if} \quad t>1/\beta\end{cases}$$

profit maximizing choices in each of the above three cases

$$Y=\begin{cases}0 & \text{if} \quad p<AC \\ \geq 0 & \text{if} \quad p=AC \\ \text{NONE} & \text{if} \quad p>AC\end{cases}$$

answer to problem 5,case $\alpha>1,\beta>1$

Cost minimizing choices

$$[K,L]=\begin{cases}[0,Y] & \text{if} \quad t<1/\beta \\ [\frac{Y(\alpha-1)}{\alpha\beta-1},\frac{Y(\beta-1)}{\alpha\beta-1}] & \text{if} \quad 1/\beta<t<\alpha \\ [Y,0] & \text{if} \quad t>\alpha\end{cases}$$

$$AC=\begin{cases}w & \text{if} \quad t<1/\beta \\ r\frac{(\alpha-1)}{\alpha\beta-1}+w\frac{(\beta-1)}{\alpha\beta-1} & \text{if} \quad 1/\beta<t<\alpha \\ r & \text{if} \quad t>\alpha\end{cases}$$

profit maximizing choices in each of the above three cases

$$Y=\begin{cases}0 & \text{if} \quad p<AC \\ \geq 0 & \text{if} \quad p=AC \\ \text{NONE} & \text{if} \quad p>AC\end{cases}$$

answer to problem 5,case $\alpha \geq 1, \beta \leq 1$

Cost minimizing choices

$$[K,L] = \begin{cases} [0,Y] & \text{if} \quad t < 1/\beta \\ [\beta K + L = Y] & \text{if} \quad t = 1/\beta \\ [Y/\beta, 0] & \text{if} \quad t > 1/\beta \end{cases}$$

$$AC = \begin{cases} w & \text{if} \quad t < 1/\beta \\ w = r/\beta & \text{if} \quad t = 1/\beta \\ r/\beta & \text{if} \quad t > 1/\beta \end{cases}$$

profit maximizing choices in each of the above three cases

$$Y = \begin{cases} 0 & \text{if} \quad p < AC \\ \geq 0 & \text{if} \quad p = AC \\ \text{NONE} & \text{if} \quad p > AC \end{cases}$$

answer to problem 5,case $\alpha \leq 1, \beta \geq 1$

Cost minimizing choices

$$[K,L] = \begin{cases} [0,Y/\alpha] & \text{if} \quad t < \alpha \\ [K + \alpha L = Y] & \text{if} \quad t = \alpha`` \\ [Y, 0] & \text{if} \quad t > \alpha \end{cases}$$

$$AC = \begin{cases} w/\alpha & \text{if} \quad t < \alpha \\ w/\alpha = r & \text{if} \quad t = \alpha \\ r & \text{if} \quad t > \alpha \end{cases}$$

profit maximizing choices in each of the above three cases

$$Y = \begin{cases} 0 & \text{if} \quad p < AC \\ \geq 0 & \text{if} \quad p = AC \\ \text{NONE} & \text{if} \quad p > AC \end{cases}$$

# PROBLEM 6

**find all global maxima of the following maximization problem, or show that none exist**

$$\boxed{\begin{array}{l} \text{maximization problem} \\ \text{Objective function } \mathbb{R}^n \xrightarrow{f} \mathbb{R} \\ f(x) = px = p_1x_1 + \ldots + p_nx_n \\ \text{feasible set } \quad S \subseteq \mathbb{R}^n \\ \text{Variables } \quad x \in \mathbb{R}^n \\ \text{Parameters } \quad p \in \mathbb{R}^n \\ \text{conditions on parameters:} \\ \text{there exists } b \in S \text{ such that } pb > 0 \\ \text{if } x \in S \text{ and } t \geq 1 \text{ then } tx \in S \end{array}} \quad (6)$$

No global max exists

## PROBLEM 7

**find all global maxima of the following maximization problem, or show that none exist**

$$\boxed{\begin{array}{l} \text{maximization problem} \\ \text{Objective function } \quad \mathbb{R}^2 \xrightarrow{f} \mathbb{R} \\ f(x,y) = \exp(-x^2 + xy - y^2 + x + y - 1) \\ \text{Constraints } \quad x \geq 0, y \geq 2 \\ \text{Variables } \quad x, y \\ \text{Parameters } \quad \text{none} \end{array}}$$

$G = -x^2 + xy - y^2 + x + y - 1, f = \exp(G)$ ,so we must maximize G

$$\boxed{\begin{array}{l} \text{answer to problem 7} \\ x = 3/2, y = 2 \end{array}}$$

## PROBLEM 8

**find all global maxima of the following maximization problem, or show that none exist**

| maximization problem | |
|---|---|
| Objective function | $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ |
| $f(x,y) = \dfrac{xy + x + y - x^2 - y^2}{x^2 + y^2 - xy - x - y + 2}$ | |
| Constraints | $x \geq 0, y \geq 2$ |
| Variables | $x, y$ |
| Parameters | none |

(7)

Let $N = xy + x + y - x^2 - y^2, D = x^2 + y^2 - xy - x - y + 2$. Then

$N + D = 2, f = \dfrac{N}{D} = \dfrac{2-D}{D} = \dfrac{2}{D} - 1$ ,and therefore to maximize f we need to minimize D

| answer to problem 8 |
|---|
| $x = 3/2, y = 2$ |