PROBLEM 1

Let $[a,b]$ be a basis of R^2 ,and let $\alpha \in R, \beta \in R$ be two real numbers. For which values of α, β is $[a+b, \alpha a]$ a basis of R^2 ? For which values of α, β is $[\alpha a, \beta b]$ a basis of R^2 ?

Answers: $\alpha \neq 0, \beta \neq 0$

PROBLEM 2

Consider the following subset of R^5

$$W = \{x \in R^5 : x_1 - x_3 - x_5 = 0\} \quad (1)$$

1.show that W is closed under linear combinations, hence a subspace of R^5

2.Find a linear map $R^5 \xrightarrow{T} R^5$ such that $W = \text{nullspace}(T)$ .Can this linear map T be one-to-one? onto?

3.Find a linear map $R^5 \xrightarrow{T} R^5$ such that $W = \text{Range}(T)$ .Can this linear map T be one-to-one? onto?

4. Find a basis of W

Answers:

A basis of W is $\beta = [e_2, e_1 + e_3, e_4, e_1 + e_5]$ .It extends to a basis $\gamma = \beta \cup \{e_1\}$.

A linear map with $W = \text{range}(T)$ is defined by $T(e_1) = 0, T(x) = x, \forall x \in \beta$ .No such map can be one-to-one, hence by the dimension theorem it cannot be onto either.

A linear map with $W = \text{nullspace}(T)$ is defined by $T(e_1) = e_1, T(x) = 0, \forall x \in \beta$ .No such map can be onto, hence by the dimension theorem it cannot be one-to-one either.

PROBLEM 3

Consider the following subset of R^5

$$A = \{x \in R^5 : x_2 - x_5 = 4\} \quad (2)$$

1.show that A is closed under affine combinations, hence a flat in R^5 .Is it a hyperplane?

2.Find an affine map $R^5 \xrightarrow{T} R^5$ such that $A = \text{nullspace}(T)$ .Can this affine map T be one-to-one? onto?

3.Find an affine map $R^5 \xrightarrow{T} R^5$ such that $A = \text{Range}(T)$ .Can this affine map T be one-to-one? onto?

Answers

1.A is a hyperplane because it is defined by a single linear equation

2.choose any element of A, say $b = 4e_2$.The subspace uniquely defined by the hyperplane A is $W = A - b = \{x \in R^5 : x_2 = x_5\}$.A basis of W is $\beta = [e_1, e_2 + e_5, e_3, e_4]$,and $\gamma = \beta \cup \{e_2\}$ is a basis of R^5 extending β .The linear map $R^5 \xrightarrow{L} R^5$ uniquely defined by $L(e_2) = e_2, L(x) = 0, \forall x \in \beta$ satisfies $W = \text{nullspace}(L)$ .The affine map $R^5 \xrightarrow{T} R^5, T(x) = L(x) - L(b) = L(x) - b$ satisfies $A = \text{nullspace}(T)$,because $T(x) = 0 \Leftrightarrow x - b \in \text{nullspace}(L) = W \Leftrightarrow x \in A$.

For <u>any</u> affine map $R^5 \xrightarrow{F} R^5, F(x) = M(x) + c, M$ linear, that satisfies $A = \text{nullspace}(F)$, we have $F(4e_2) = 0 = F(5e_2 + e_5)$,hence F, and M, cannot be one-to-one. By the dimension theorem M cannot be onto, hence F cannot be onto either.

3. The linear map $R^5 \xrightarrow{L} R^5$ uniquely defined by $L(e_2) = 0, L(x) = x, \forall x \in \beta$ satisfies $W = \text{range}(L)$ .The affine map $R^5 \xrightarrow{T} R^5, T(x) = L(x) + b$ satisfies $A = \text{range}(T)$,because $y = T(x) \Leftrightarrow y = b + L(x) \in b + W = A$.

By the same reasoning, no such affine map can be either one to one or onto

PROBLEM 4

Consider the following subset of R^5

$$C = \{x \in R^5 : x_2 - x_5 \geq 0, x_1 - x_2 \leq 0, x_3 \geq 0\} \quad (3)$$

1.show that C is closed under nonnegative linear combinations, hence a convex cone in R^5

2.Find a linear map $R^5 \xrightarrow{T} R^5$ such that $C = \{x \in R^5 : T(x) \geq 0\}$ .Can this linear map T be one-to-one? onto?

3.show that C is an intersection of half-spaces through the origin. Describe these half-spaces explicitly

Answers: one such map is $R^5 \xrightarrow{T} R^5$ given by

$$T(x) = [x_2 - x_5, x_2 - x_1, x_3, 0, 0] \quad (4)$$

The vectors $x_t = [t, t, 0, 0, t] = t(e_1 + e_2 + e_5)$ belong to C for all t. Hence any linear map T that satisfies $C = \{x \in R^5 : T(x) \geq 0\}$ must also satisfy

$$tT(e_1 + e_2 + e_5) \geq 0, \forall t \in R \quad (5)$$

Inequality (5) for t=1 yields

$$T(e_1 + e_2 + e_5) \geq 0 \quad (6)$$

Inequality (5) for t=-1 yields

$$T(e_1 + e_2 + e_5) \leq 0 \quad (7)$$

By (6) and (7)

$$T(e_1 + e_2 + e_5) = 0 = T(0) \quad (8)$$

Hence any linear map T that satisfies $C = \{x \in R^5 : T(x) \geq 0\}$ cannot be either one to one or onto.

PROBLEM 5

Consider the following subset of R^4

$$C = \{x \in R^4 : x_2 - x_4 \geq 6, x_1 - x_2 - x_3 \leq 7, x_3 \geq 0\} \quad (9)$$

1.show that C is closed under convex combinations, hence a convex set in R^5

2.Find an affine map $R^4 \xrightarrow{T} R^4$ such that $C = \{x \in R^4 : T(x) \geq 0\}$ .Can this affine map T be one-to-one? onto?

3.show that C is an intersection of half-spaces. Describe these half-spaces explicitly

Answers: one such map is $R^4 \xrightarrow{T} R^4$ given by

$$\begin{aligned} T(x) &= L(x) - b \\ L(x) &= [x_2 - x_4, x_2 + x_3 - x_1, x_3, 0] \\ b &= [6, -7, 0, 0] \end{aligned} \quad (10)$$

The vectors $x_t = t(e_1 + e_2 + e_4) + w, t \in R, w = [13, 6, 0, 0]$ belong to C for all t. Hence any affine map $T'(x) = L'(x) - b'$ that satisfies $C = \{x \in R^5 : T'(x) \geq 0\}$ must also satisfy $T'(x_t) \geq 0, \forall t \in R$,hence $L'(t(e_1 + e_2 + e_4) + w) - b' \geq 0, \forall t \in R$

$$tL'(e_1 + e_2 + e_4) \geq b' - L'(w), \forall t \in R \tag{11}$$

By (11) we obtain

$$L'(e_1 + e_2 + e_4) \geq \frac{b' - L'(w)}{t}, \forall t > 0 \tag{12}$$

$$L'(e_1 + e_2 + e_4) \leq \frac{b' - L'(w)}{t}, \forall t < 0 \tag{13}$$

By (12),(13)

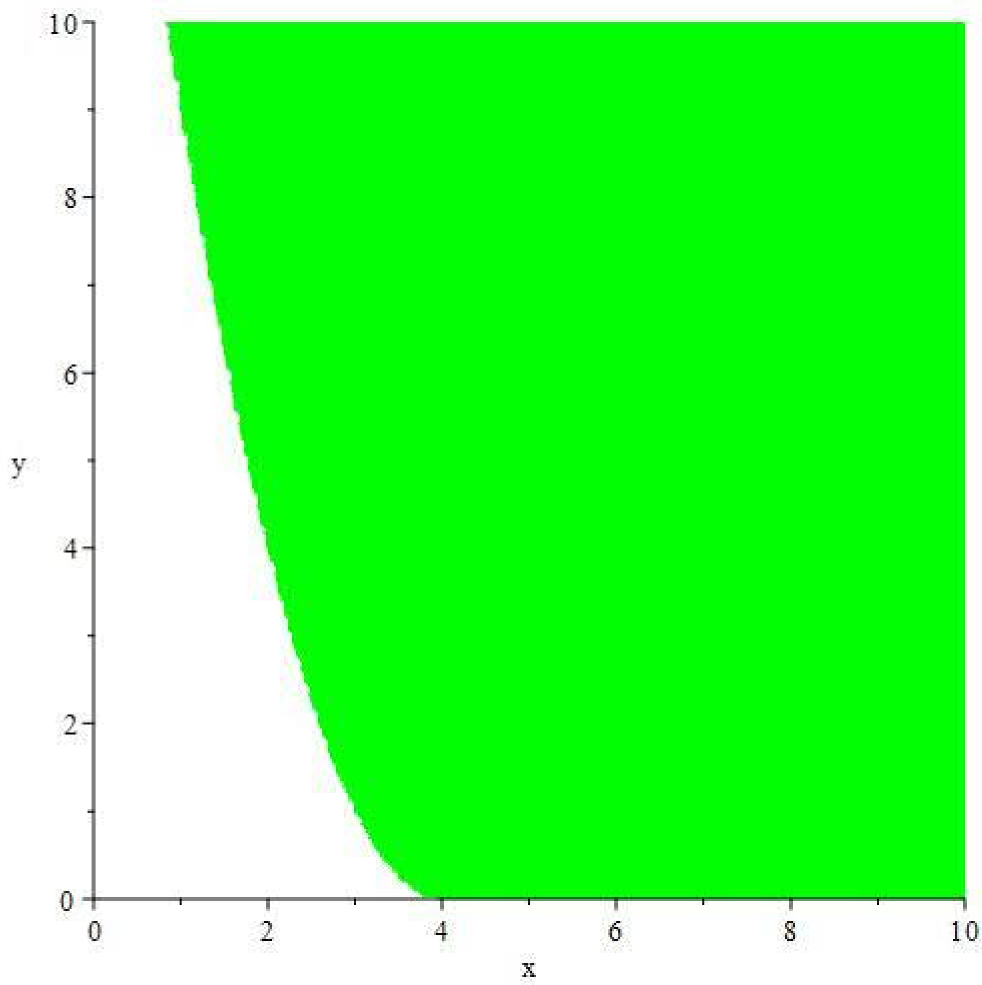
$$L'(e_1 + e_2 + e_4) = 0 = L'(0) \tag{14}$$

Hence L', T' cannot be either one to one or onto.

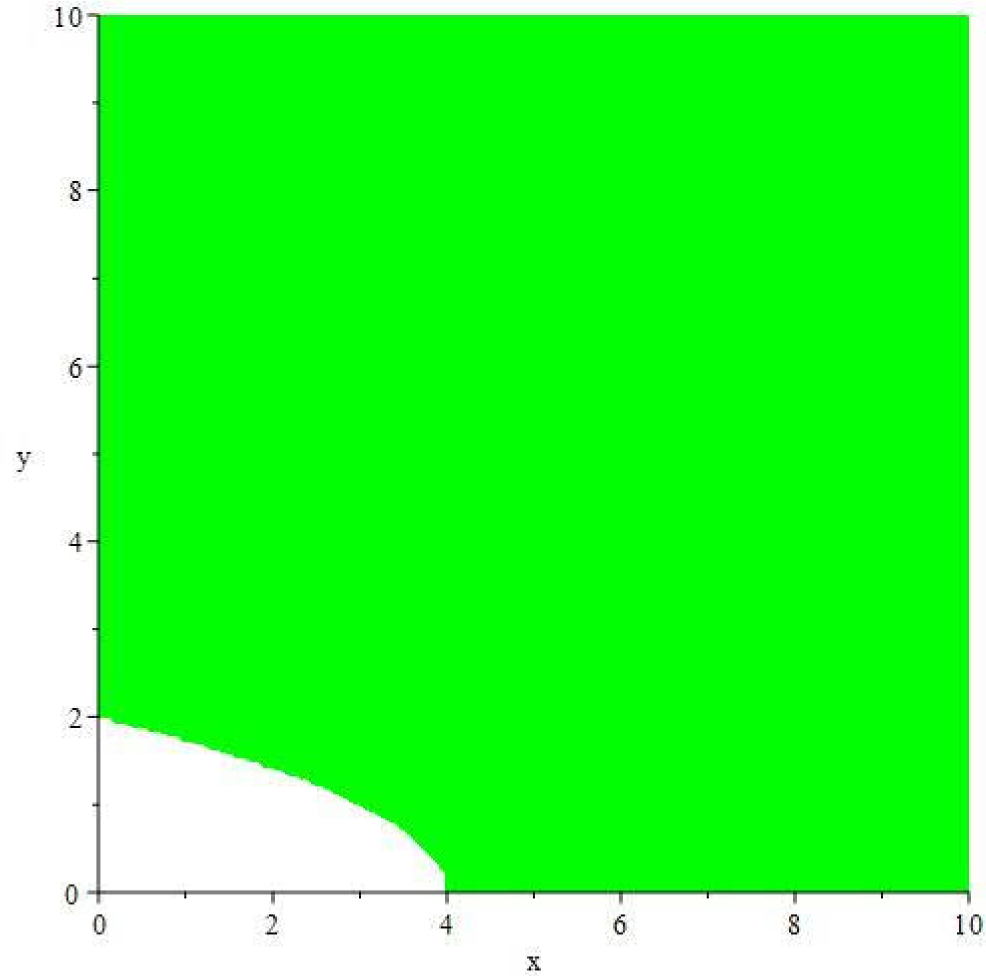
PROBLEM 6

For each one of the following functions f ,and for each value of the real parameter c, compute and draw their better-than sets $B_c^f = \{(x,y) \in R_+^2 : f(x,y) \geq c\}$; state whether they are quasi-concave functions on R_+^2 ,or concave functions on R_+^2

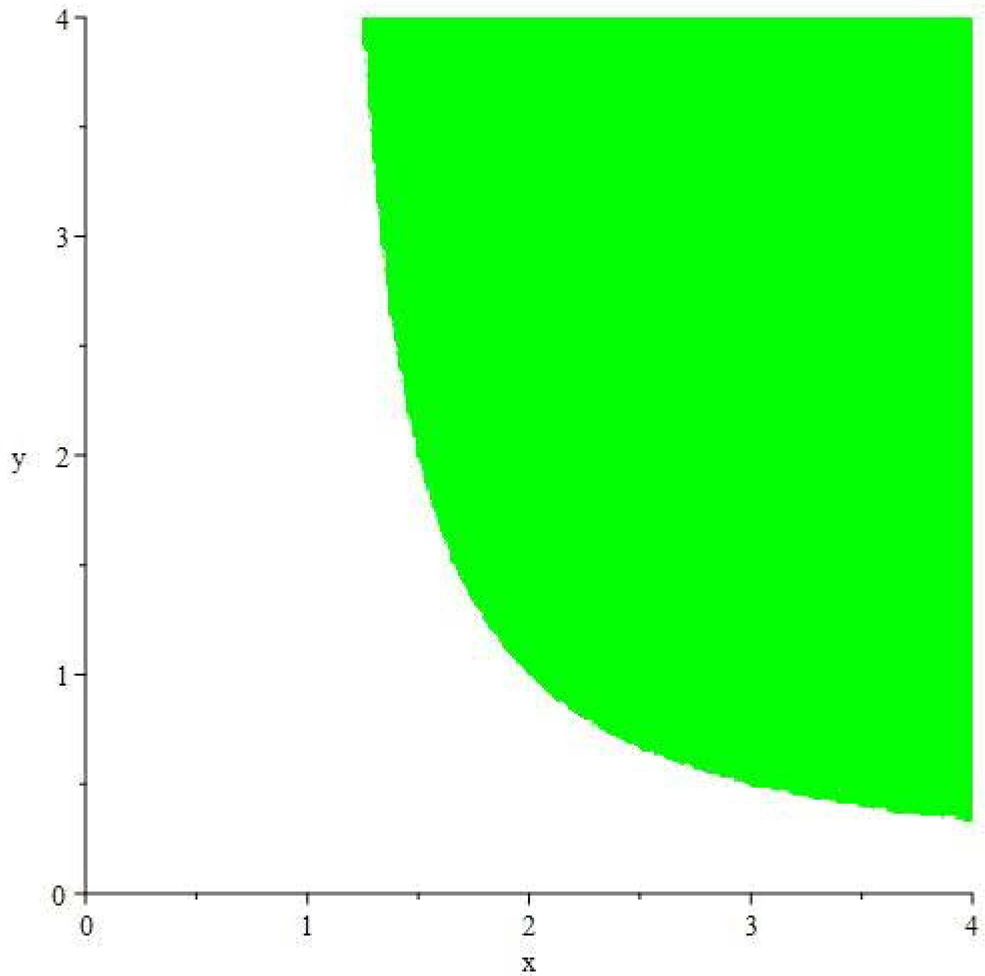
- $f(x,y) = x + \sqrt{y}, c = 4$;f is concave



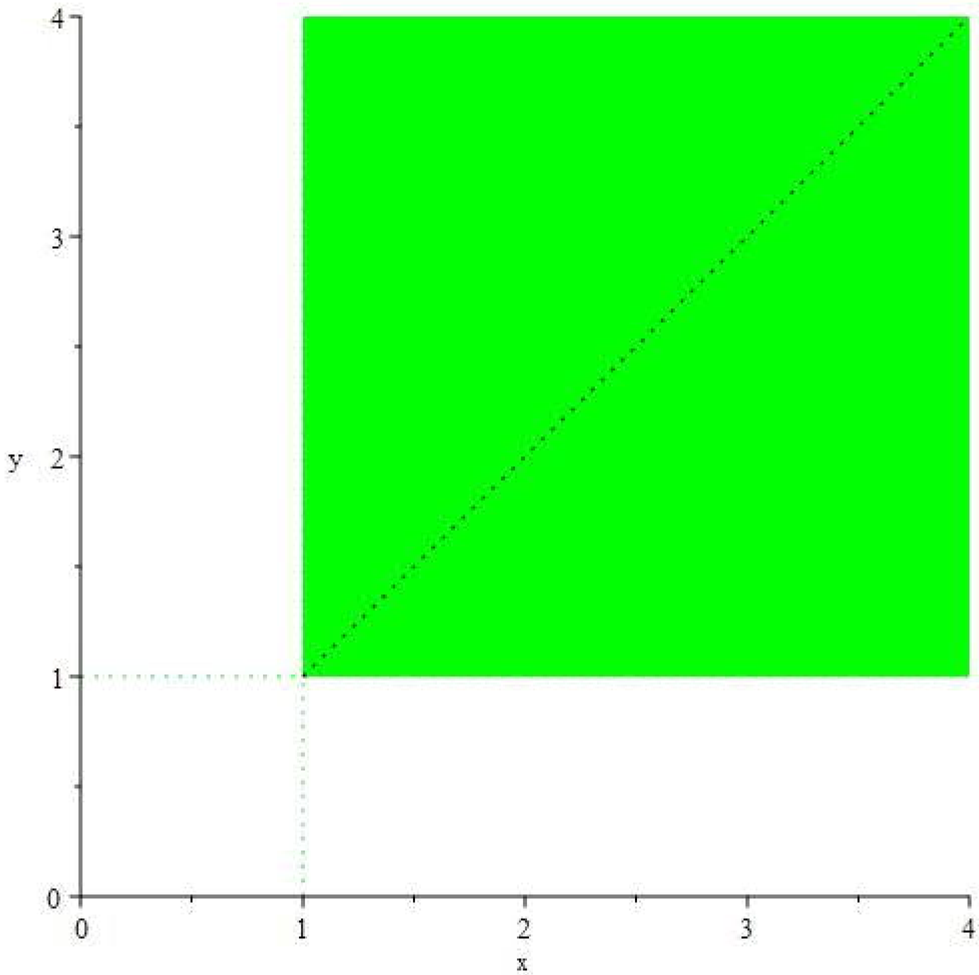
- $f(x,y) = x + y^2, c = 4$; f is not quasi-concave



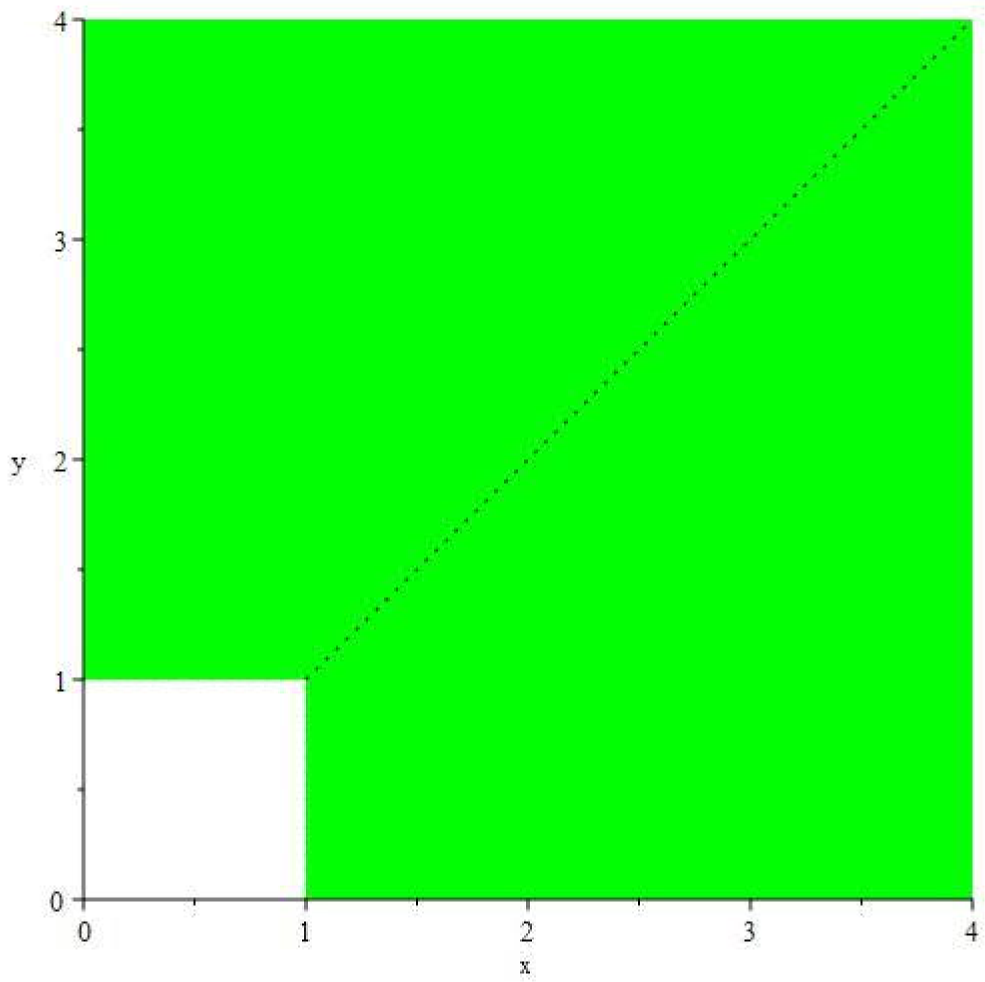
- $f(x,y) = x - \frac{1}{y}, c = 4$;f is concave



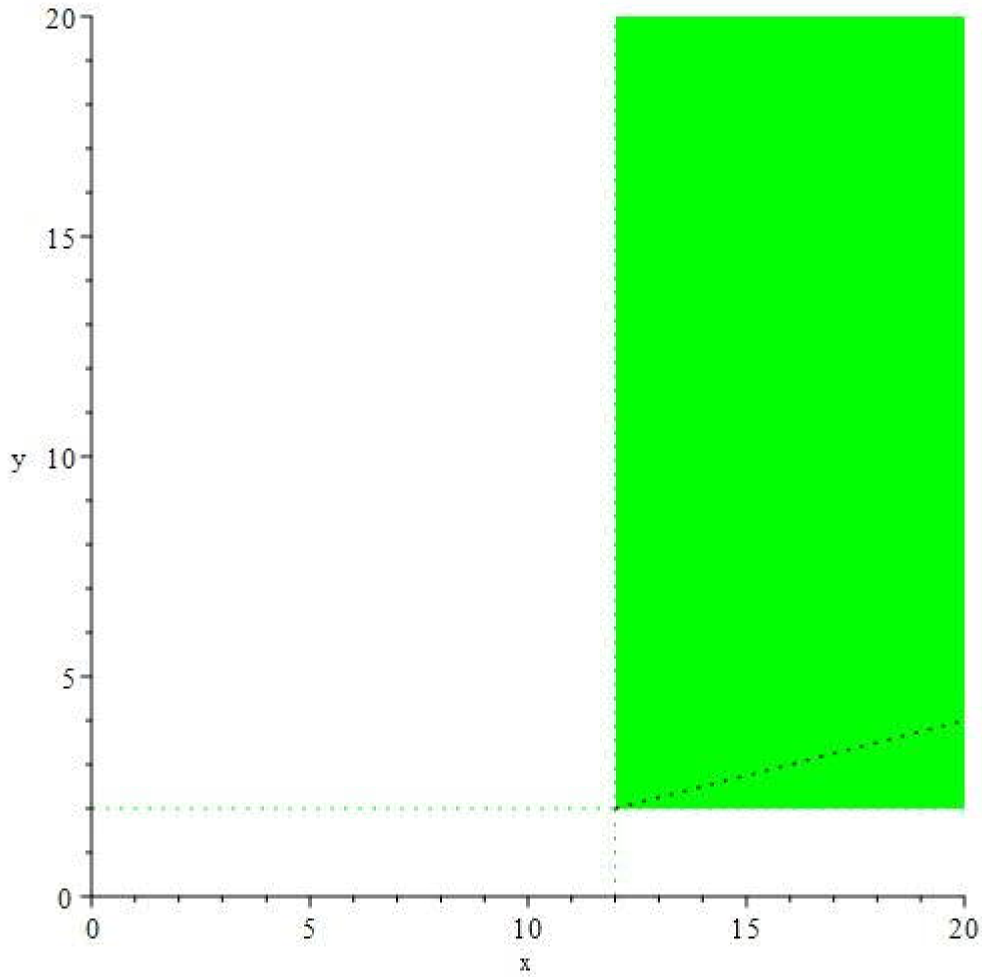
- $f(x, y) = \min(x, y), c = 1$;f is concave



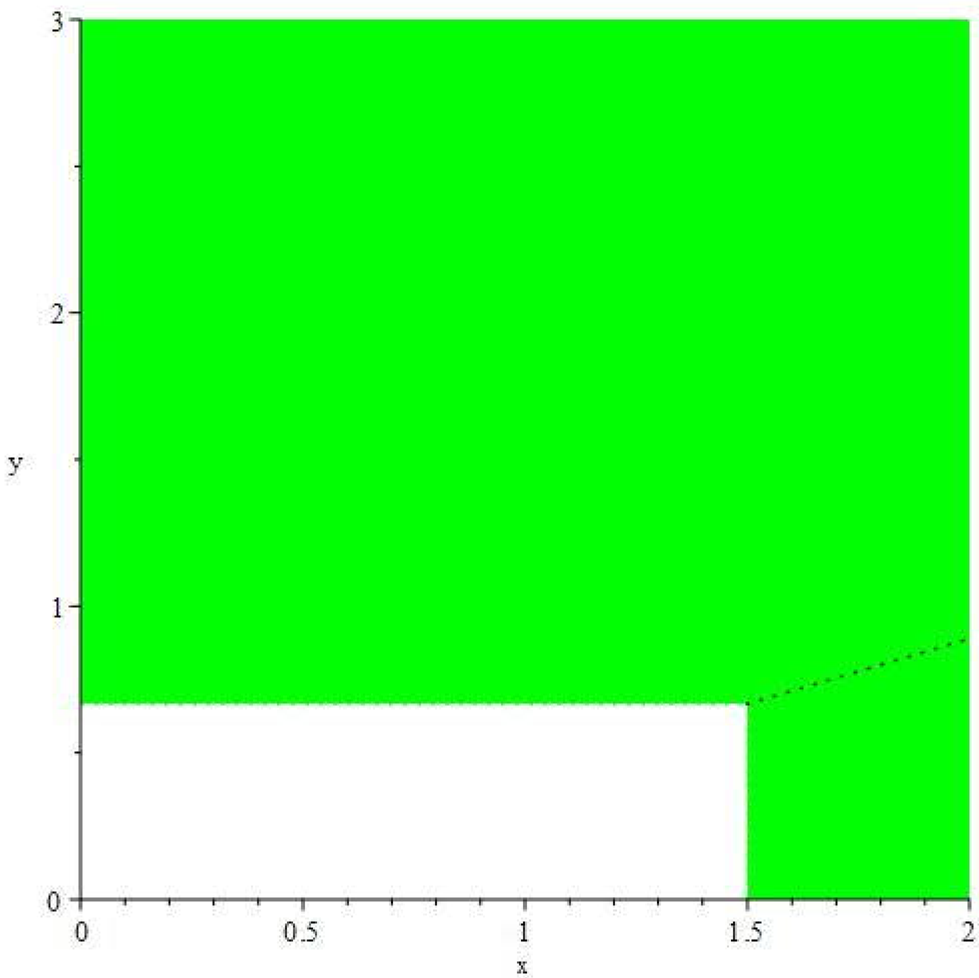
- $f(x, y) = \max(x, y), c = 1$;f is not quasi-concave



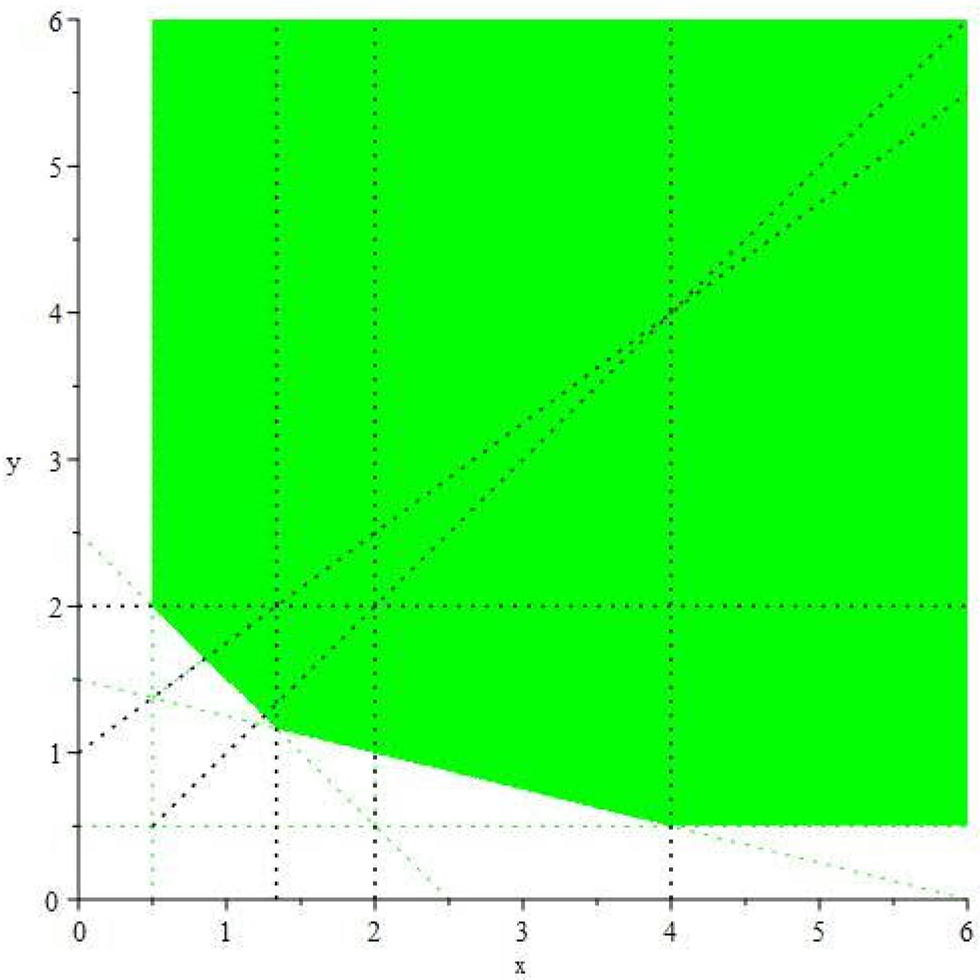
- $f(x,y)=\min(x/4+1, y+2), c=3$; f is concave

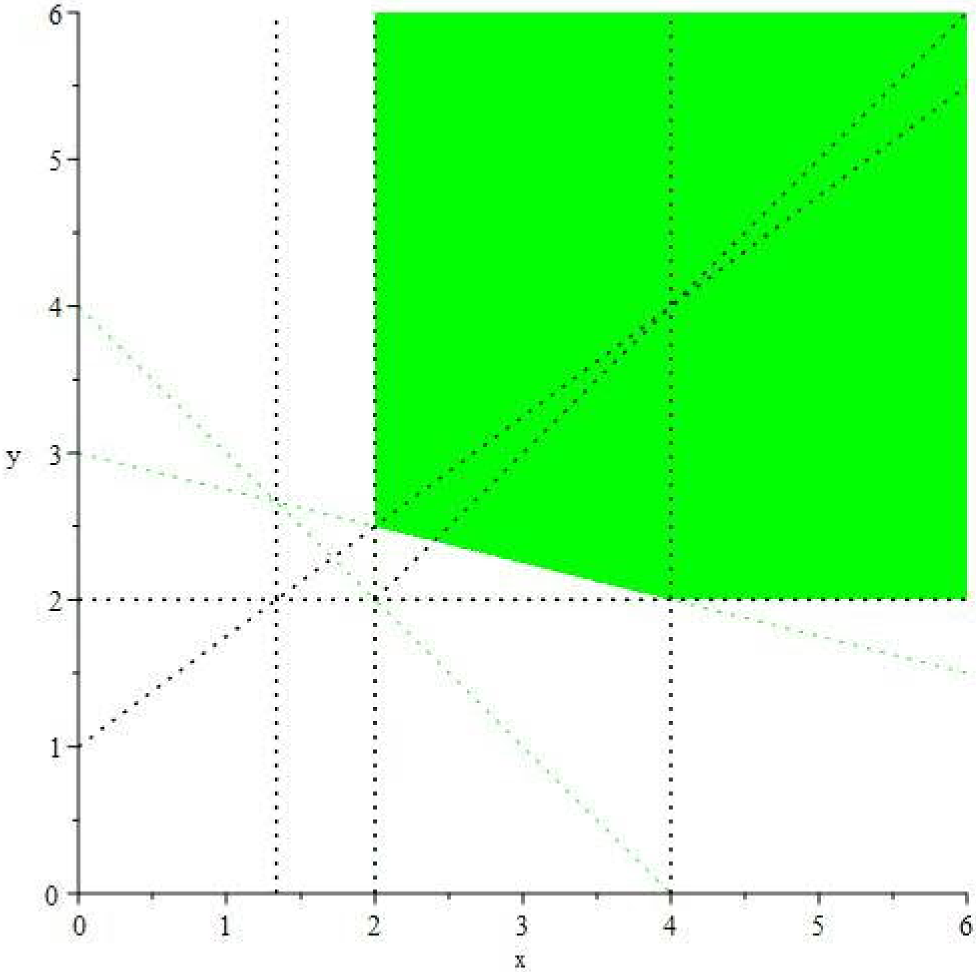


- $f(x,y)=\max(2x/3,3y/2), c=1$;f is not quasi-concave



- $f(x,y) = \min(x/4 + y - 1, x + y - 2, x, y), c = 1/2, 2, 4$;f is concave





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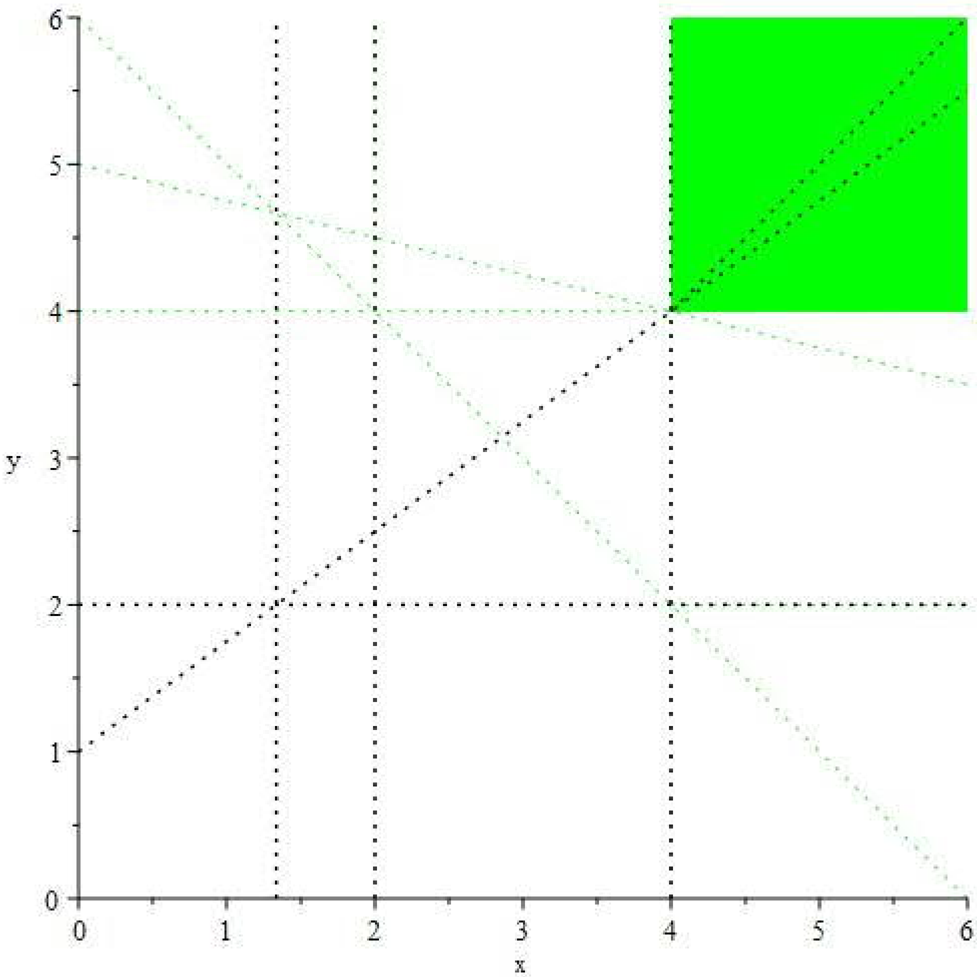

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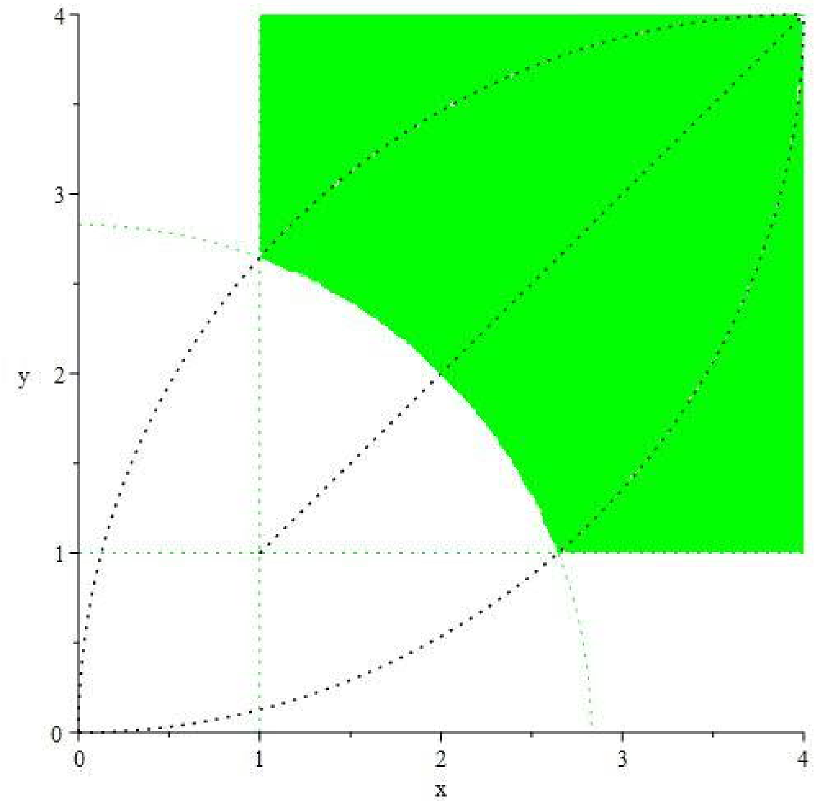
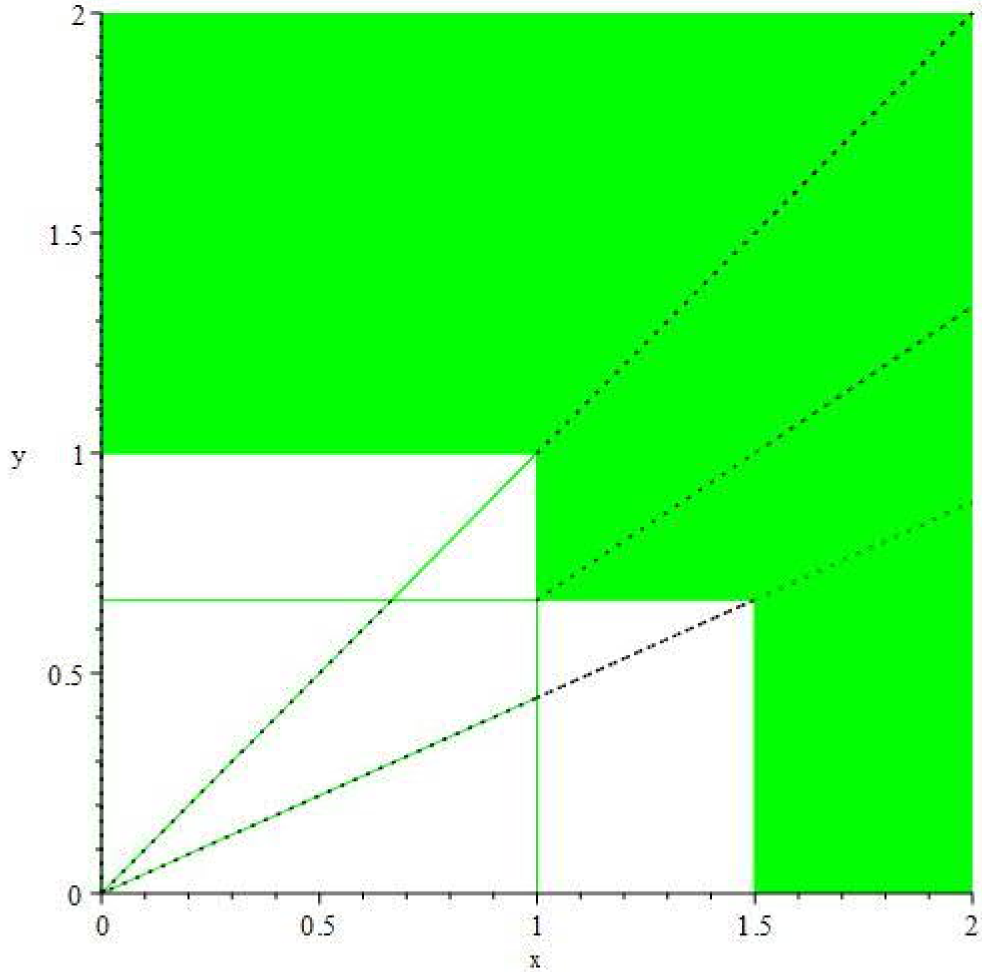
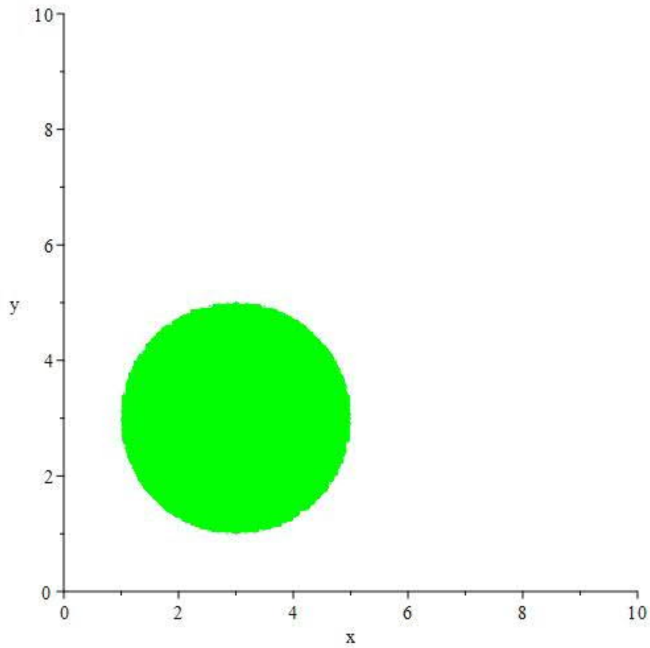
- $f(x,y)=\min(x,y,\frac{x^2+y^2}{8}), c=1$;f is not quasi-concave



- $f(x,y)=\min(\max(x,y),\max(2x/3,3y/2)), c=1$;f is not quasi-concave



- $f(x,y)=-(x-3)^2-(y-3)^2, c=-4$; f is concave



PROBLEM 7

Find all global maxima of the following maximization problem, or show that none exist

Objective function $f(x) = 3(2\sqrt{x+1} - 2) - 9x$

constraints $x \geq 0$

variables x

answer: $x = 0$

PROBLEM 8

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $f(x_1, x_2, x_3) = x_1 x_2 x_3 - w_1 x_1 - w_2 x_2 - w_3 x_3$

constraints $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

variables x_1, x_2, x_3

parameters w_1, w_2, w_3

conditions on parameters $w_1 > 0, w_2 > 0, w_3 > 0$

answer: there is no global maximum, because
$f(t,t,t) = t^3 - w_1 t - w_2 t - w_3 t \to \infty$ as $t \to \infty$

PROBLEM 9

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $f(x_1, x_2) = \min(\frac{x_1}{4} + 1, x_2 + 2)$

constraints $x_1 + p x_2 \leq 4, x_1 \geq 0, x_2 \geq 0$

variables x_1, x_2

parameters p .conditions on parameters $p > 0$

answer: $x_1 = 4, x_2 = 0$