TAKEHOME EXAM: ANSWERS

Please answer all questions. Omitting calculations is OK. Recall that the set of efficient allocations depends **only** on preferences, technologies, and aggregate endowments

QUESTION 1

THE ECONOMY

- Two goods, A and X ,written in this order.
- Two consumers,1 and 2.
- One firm.

Consumer 1

- Consumption set R_+^2
- Endowment vector $\omega_1 = [0,4]$
- Profit share $\theta_1 = 0$
- Utility function $u_1 = \log A_1 + \log X_1$

Consumer 2

- Consumption set R_+^2
- Endowment vector $\omega_2 = [0,0]$
- Profit share $\theta_2 = 1$
- Utility function $u_2 = X_2$

The firm produces good A out of good X with technology described by the production function

$$A = \sqrt{2X}$$

Policy: The firm's profit Π is taxed at a rate t, i.e. after-tax profit is $(1-t)\Pi$. Consumer 1 receives a lump-sum transfer T

Parameters: t, T .**Conditions on parameters**: $0 \le t < 1, T \ge 0$

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all efficient allocations P . Compute and draw the Pareto frontier and the utility possibility set.
- Compute all competitive equilibria. For which values of the parameters t,T do equilibria exist? Denote the equilibrium allocation at t by $E(t)=[\left[A_i(t),X_i(t)\right]_{i=1}^{2},A(t),X(t)]$.
- First welfare theorem: For which values of the tax rate t , if any, is it true that $E(t)\subseteq P$?
- Is $P\subseteq \bigcup_{0\le t<1} E(t)$?

ANSWERS

EFFICIENT ALLOCATIONS

The set of feasible allocations

$$FA=\left\{(A_1,X_1,A_2,X_2,A,X)\in R_+^6 : A\le\sqrt{2X}, A_1+A_2\le A, X_1+X_2+X\le 4\right\} \quad (1)$$

is convex, and the objective functions u_1,u_2 are concave, hence the utility possibility set is convex, and we can compute efficient points by solving, for all values of the parameter $\alpha\in[0,1]$,the following max problem

$$\begin{aligned}&\max W=\alpha u_1+(1-\alpha)u_2\\&\text{subject to } A\le\sqrt{2X}, A_1+A_2\le A, X_1+X_2+X\le 4,\text{all variables}\ge 0\end{aligned} \quad (2)$$

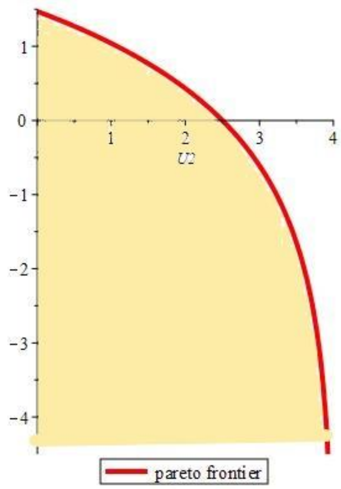
The solutions are, after eliminating α

Efficient allocations P
$A=A_1=\sqrt{2}\sqrt{X}, A_2=0, X_1=2X, X_2=4-3X, 0\le X\le\frac{4}{3}$

(3)

pareto frontier
$u_1=-\ln(3)+\frac{\ln(-3u_2+12)}{2}+\frac{\ln(2)}{2}+\ln\left(-\frac{2u_2}{3}+\frac{8}{3}\right), 0\le u_2\le 4$

(4)



EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$p =$ price of good A,w=price of good X . Normalize $w = 1$

2. DEFINE CONSUMER INCOMES

$$M_1 = 4 + T, M_2 = (1-t)\Pi \tag{5}$$

3. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

max$u_1 = \log A_1 + \log X_1$,subject to $pA_1 + X_1 \le M_1$
max $u_2 = X_2$,subject to $X_2 \le M_2$

The solutions are

$$(A_1, X_1) = \left(\frac{4+T}{2p}, \frac{4+T}{2}\right) \tag{6}$$

$$(A_2, X_2) = (0, (1-t)\Pi) \tag{7}$$

4. SOLVE THE OPTIMIZATION PROBLEMS OF FIRMS

max $(1-t)\Pi = (1-t)\left(p\sqrt{2X} - X\right), X \ge 0$

The solution is

$$(X,A,\Pi)=\left(\frac{p^2}{2},p,\frac{p^2}{2}\right) \tag{8}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$A=A_1+A_2,4=X_1+X_2+X \tag{9}$$

equilibrium with profit taxation

$$p=\frac{2\sqrt{2}}{\sqrt{4-t}},T=\frac{4t}{4-t},\Pi=\frac{4}{4-t} \tag{10}$$

Equilibrium allocation $E(t),0\le t<1$

$$A=A_1=\frac{2\sqrt{2}}{\sqrt{4-t}},A_2=0,X_1=\frac{8}{4-t},X_2=\frac{4(1-t)}{4-t},X=\frac{4}{4-t}$$

Comparing (10) to (3) we conclude that $E(t)\subseteq P,\forall t\in[0,1)$,and that $P\not\subset\bigcup_{0\le t<1}E(t)$,because $P-\bigcup_{0\le t<1}E(t)=$ efficient points with $0\le X<1$

QUESTION 2

THE ECONOMY

- Two goods, A and X ,written in this order.
- One consumer
- One firm.

The Consumer

- Consumption set R_+^2
- Endowment vector $\omega=[0,4]$
- Profit share $\theta=1$
- Utility function $u=2\sqrt{A}+2\sqrt{X}$

The firm produces good A out of good X with technology described by the production function

$$\hat{A} = \sqrt{2\hat{X}}$$

Policy: The firm receives a subsidy s per unit of output. The consumer pays a lump-sum tax T

Parameters: s,T .**Conditions on parameters**: $s \geq 0, T \geq 0$

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all efficient allocations P
- Compute all competitive equilibria. For which values of the parameters s,T do equilibria exist? Let p be the price of good A and w the price of good X. Denote the equilibrium allocation at s by $E(s) = [A(s), X(s), \hat{A}(s), \hat{X}(s)]$,the equilibrium price vector by $[p(s), w(s)]$,and the equilibrium value of profit by $\Pi(s)$.
- First welfare theorem: For which values of the subsidy s, if any, is $E(s) \subseteq P$?
- Second welfare theorem: For which values of the subsidy s, if any, are all efficient allocations decentralizable, i.e. $P \subseteq E(s)$?
- Draw the equilibrium value of utility $v(s) = u(A(s), X(s))$ against s.
- Draw the equilibrium value of the real wage $w(s)/p(s)$ against s. Is it a good measure of welfare?
- Draw the equilibrium value of total employment $\hat{X}(s)$ against s .Is it a good measure of welfare?
- Draw the equilibrium value of total output $\hat{A}(s)$ against s. Is it a good measure of welfare?
- Draw the equilibrium value of total real income $\dfrac{w(s)\hat{X}(s) + \Pi(s)}{p(s)}$ against s. Is it a good measure of welfare?

ANSWERS

EFFICIENT ALLOCATIONS

There is just one consumer, hence efficient allocations are the solutions of the following max problem

$$\max u = 2\sqrt{A} + 2\sqrt{X}$$
$$\text{subject to } A \le \hat{A} \le \sqrt{2\hat{X}}, X + \hat{X} \le 4, \text{all variables} \ge 0 \quad (11)$$

There is a unique solution, given by

$$\boxed{\begin{array}{l} \text{Efficient allocation} \\ P = \left\{\left(\hat{X}, \hat{A}, X, A\right)\right\} \\ \hat{X} = \frac{A^2}{2}, \hat{A} = A, X = A^3, A^3 + \frac{A^2}{2} = 4, A \approx 1.437 \end{array}} \quad (12)$$

The equation $A^3 + \frac{A^2}{2} = 4$ has exactly one positive real root, hence the set P of efficient allocations contains exactly one element, given by (12).

EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$p =$ price of good A,w=price of good X . Normalize $p = 1$

2. DEFINE CONSUMER INCOMES

$$M = 4w - T + \Pi \quad (13)$$

3. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

$\max u = 2\sqrt{A} + 2\sqrt{X}$,subject to $A + wX \le 4w - T + \Pi$

The solutions are

$$\left(A, X\right) = \left(\frac{wM}{\left(1+w\right)}, \frac{M}{w\left(1+w\right)}\right) \quad (14)$$

4. SOLVE THE OPTIMIZATION PROBLEMS OF FIRMS

$\max \Pi = \left(1+s\right)\sqrt{2\hat{X}} - w\hat{X}, \hat{X} \ge 0$

The solution is

$$\left(\hat{X},\hat{A},\Pi\right)=\left(\frac{(1+s)^2}{2w^2},\frac{1+s}{w},\frac{(1+s)^2}{2w}\right) \tag{15}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\hat{A}=A,\hat{X}+X=4 \tag{16}$$

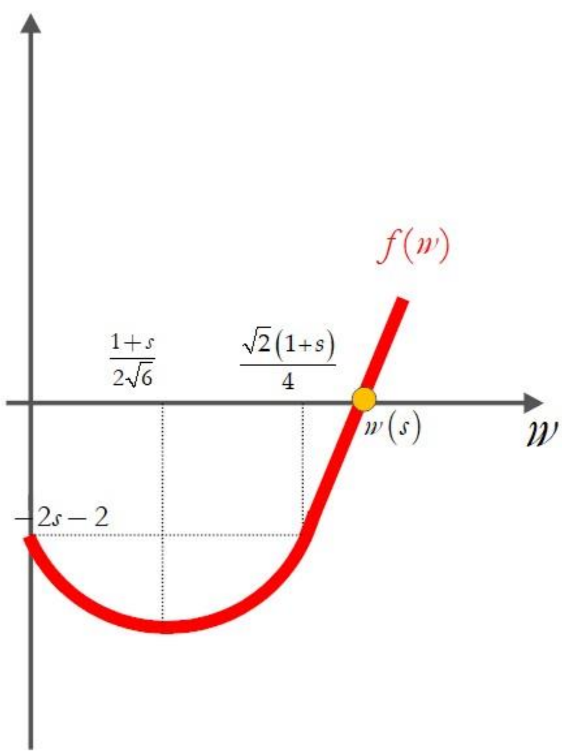
By (16),(15),(14),(13) we obtain

$$T=\frac{s^2w+8w^3-2s-w-2}{2w^2} \tag{17}$$

$$0=4-\hat{X}-X=\frac{f(w)}{2w^3},f(w)\triangleq 8w^3-(1+s)^2w-2s-2 \tag{18}$$

Equation (18) has a unique positive solution $w=w(s)$,the equilibrium real wage,

because $f(0)=f(\frac{\sqrt{2}(1+s)}{4})=-2s-2<0,f(1+s)>0,f'(w)=24w^2-(1+s)^2$



By (17),(18) we obtain $T(s)=\frac{s(s+1)}{w(s)}$,and that the set $E(s)$ of equilibrium allocations contains exactly one element for each $s\geq 0$,and therefore $E(s)\subseteq P$ iff $E(s)=P$ iff $P\subseteq E(s)$.

$$\boxed{\begin{array}{l}\text{unique equilibrium allocation}\\ w=w(s)\text{ is the unique positive solution of }8w^3-(1+s)^2w-2s-2=0\\ E(s)=\left\{\left(\hat{X}(s),\hat{A}(s),X(s),A(s)\right)\right\}\\ \hat{X}(s)=\frac{(1+s)^2}{2w(s)^2},A(s)=\hat{A}(s)=\frac{1+s}{w(s)},X(s)=4-\frac{(1+s)^2}{2w(s)^2}\end{array}} \quad (19)$$

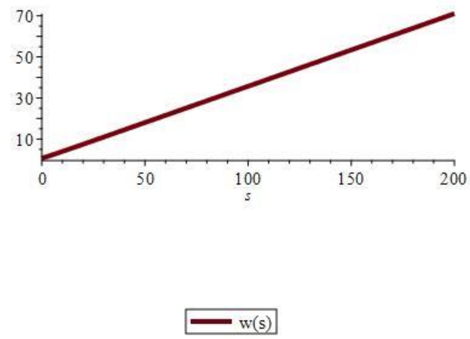
By (19),(12) we obtain that $P=E(s)$ only if $X(s)=\left(A(s)\right)^3$ .From (19) we obtain

$X(s)-\left(A(s)\right)^3=-\frac{s^3+3s^2+2s}{w(s)^3}<0$, unless s=0.Hence

$$\boxed{\begin{array}{l}s>0\Leftrightarrow P\cap E(s)=\varnothing,E(s)\not\subset P,P\not\subset E(s)\\ s=0\Leftrightarrow P=E(s)\end{array}} \quad (20)$$

The effect of the subsidy s on the real wage $w(s)$ is obtained by differentiating (18) w.r.t. s; $f\left(w(s),s\right)=0\ \forall s\geq 0$ implies $\frac{\partial f}{\partial w}\frac{\partial w}{\partial s}+\frac{\partial f}{\partial s}=0$,i.e.

$$\frac{\partial(\text{real wage})}{\partial s}=-\frac{\partial f/\partial s}{\partial f/\partial w}=2\frac{(1+s)w+1}{24w^2-(1+s)^2}>0 \quad (21)$$



The effect of the subsidy s on the output $A(s)$ is obtained by differentiating

$A(s)=\dfrac{1+s}{w(s)}$ w.r.t. s; $\dfrac{dA}{ds}=\dfrac{\partial A}{\partial s}+\dfrac{\partial A}{\partial w}\dfrac{\partial w}{\partial s}$ yields

$$\frac{d(\text{output})}{ds}=\frac{4(s+1)}{w^2\left(24w^2-(s+1)^2\right)}>0 \tag{22}$$

The effect of the subsidy s on employment $\hat{X}(s)$ is obtained by differentiating

$\hat{X}(s)=\dfrac{(A(s))^2}{2}$ w.r.t. s;

$$\frac{d(\text{employment})}{ds}=\frac{4(1+s)^2}{\left(24w^2-(1+s)^2\right)w^3}>0 \tag{23}$$

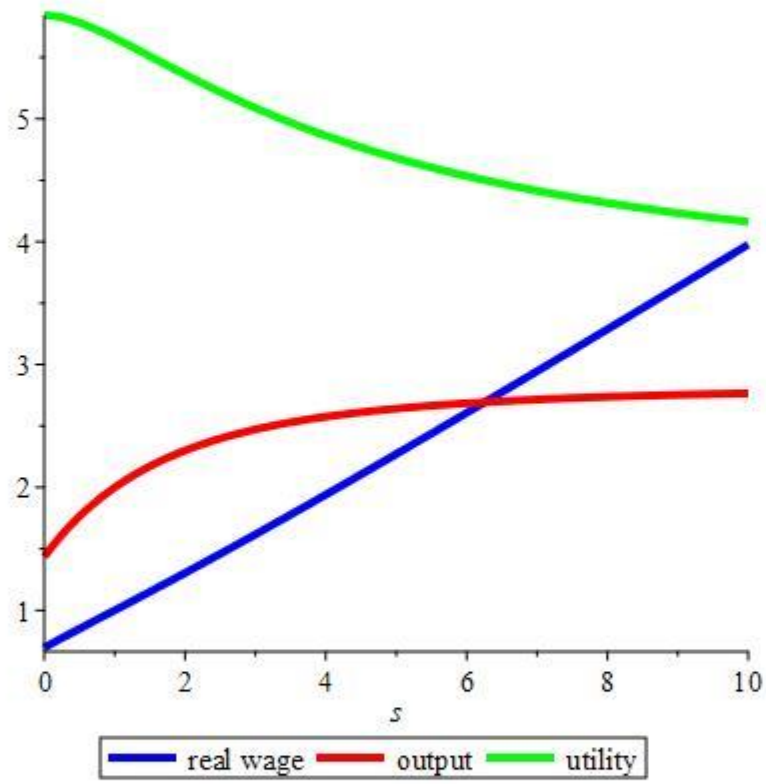
The effect of the subsidy s on total real income $w(s)\hat{X}(s)+\Pi(s)=(1+s)A(s)=\dfrac{(1+s)^2}{w(s)}$

is obtained by differentiating it w.r.t. s;

$$\frac{d(\text{real income})}{ds}=\frac{2(1+s)^2(sw+w+5)}{w^2\left(24w^2-(1+s)^2\right)}>0 \tag{24}$$

The effect of the subsidy s on equilibrium utility $v(s) = u(A(s), X(s)) = \dfrac{2\sqrt{\dfrac{1+s}{w}}(1+w)}{w}$ is obtained by differentiating it w.r.t. s;

$$\frac{d(\text{utility})}{ds} = -\frac{4s(1+s)}{\sqrt{\dfrac{1+s}{w}}w^2\left(24w^2 - (1+s)^2\right)} < 0 \tag{25}$$



QUESTION 3

THE ECONOMY

- Two goods, A and X,written in this order.
- One consumer
- One firm.

The Consumer

- Consumption set R_+^2
- Endowment vector $\omega = [0, \bar{X}]$
- Profit share $\theta = 1$
- Utility function $u = \alpha \log A + (1-\alpha)\log X$

The firm produces good A out of good X with technology described by the production function

$$\hat{A} = \begin{cases} 0 & \text{if} \quad \hat{X} \le F \\ \sqrt{2\left(\hat{X} - F\right)} & \text{if} \quad \hat{X} \ge F \end{cases}, \hat{X} \ge 0$$

Parameters: $\alpha, F, \bar{X}$.**Conditions on parameters**: $0 < \alpha < 1, 0 \le F < \bar{X}$

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all competitive equilibrium allocations E
- Compute all efficient allocations P
- First welfare theorem: For which parameter values, if any, is it true that $E \subset P$?
- Compute the set of decentralizable efficient allocations $P \cap E$
- Second welfare theorem: For which parameter values, if any, is it true that all efficient allocations are decentralizable, i.e. that $P = E$?

ANSWERS

EQUILIBRIA

$$\begin{array}{|l|}\hline \underline{\alpha\bar{X} < 2F|} \Rightarrow E = \varnothing \\ \underline{\alpha\bar{X} \ge 2F|} \Rightarrow \dfrac{p}{w} = \sqrt{\dfrac{2\alpha}{2-\alpha}}\sqrt{\bar{X}-F}, E = \left\{\left(A, \hat{A}, X, \hat{X}\right)\right\}, \\ A = \hat{A} = \sqrt{\dfrac{2\alpha}{2-\alpha}}\sqrt{\bar{X}-F}, \hat{X} = \dfrac{\bar{X}\alpha + 2F\left(1-\alpha\right)}{2\text{-}\alpha}, X = \dfrac{2\left(1\text{-}\alpha\right)\left(\bar{X}-F\right)}{2\text{-}\alpha} \\ \hline \end{array} \quad (26)$$

EFFICIENT ALLOCATIONS

Efficient allocations are the solutions of the following max problem

$$\max u = \alpha \log A + (1-\alpha)\log X$$
$$\text{subject to } A \le \hat{A}, X + \hat{X} \le \bar{X}, \text{all variables } \ge 0, \text{and} \tag{27}$$
$$\hat{X} = \begin{cases} 0 & \text{if } \hat{A} = 0 \\ F + \dfrac{\hat{A}^2}{2} & \text{if } \hat{A} > 0 \end{cases}$$

The unique solution is

$$\boxed{P = \left\{\left(A, \hat{A}, X, \hat{X}\right)\right\}, A = \hat{A} = \sqrt{\frac{2\alpha}{2-\alpha}}\sqrt{\bar{X} - F}, \hat{X} = \frac{\bar{X}\alpha + 2F(1-\alpha)}{2\text{-}\alpha}, X = \frac{2(1\text{-}\alpha)(\bar{X} - F)}{2\text{-}\alpha}} \tag{28}$$

By (28),(26) we conclude that

$$\boxed{\begin{array}{l} \alpha\bar{X} < 2F \Rightarrow E = \varnothing \subset P \ne \varnothing, P \not\subset E \\ \alpha\bar{X} \ge 2F \Rightarrow P = E \end{array}} \tag{29}$$

QUESTION 4

THE ECONOMY

- Two goods, 1 and 2, written in this order.
- Two consumers, A and B

Consumer A

- Consumption set R_+^2
- Endowment vector $\omega_A = [\alpha_1, \alpha_2]$
- Utility function $u_A = A_1$

Consumer B

- Consumption set R_+^2
- Endowment vector $\omega_B = [1-\alpha_1, 1-\alpha_2]$
- Utility function $u_B = B_1 + B_2$

Parameters: $\omega_A = [\alpha_1, \alpha_2]$.**Conditions on parameters:** $0 < \alpha_i < 1, i = 1,2$

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all efficient allocations P .Compute and draw the Pareto frontier and the utility possibility set.
- Compute all competitive equilibrium allocations $E(\omega_A)$
- First welfare theorem: For which values of ω_A ,if any, is it true that $E(\omega_A) \subseteq P$?
- Compute the set of decentralizable efficient allocations $P \cap \left(\bigcup_{\omega_A} E(\omega_A) \right)$. The union is taken over all allowed values of ω_A
- Second welfare theorem: are all efficient allocations decentralizable, i.e. is $P \subseteq \bigcup_{\omega_A} E(\omega_A)$?

ANSWERS

EFFICIENT ALLOCATIONS

The set of feasible allocations

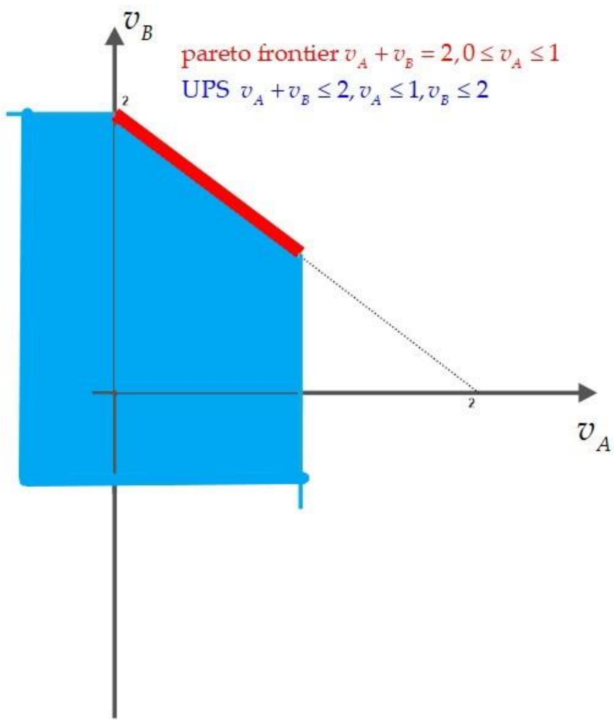
$$FA = \left\{ (A_1, A_2, B_1, B_2) \in R_+^4 : A_1 + A_2 \le 1, B_1 + B_2 \le 1 \right\} \tag{30}$$

is convex, and the objective functions u_A, u_B are concave, hence the utility possibility set is convex, and we can compute efficient points by solving, for all values of the parameter $\alpha \in [0,1]$,the following max problem

$$\begin{aligned} &\max W = \alpha u_A + (1-\alpha) u_B \\ &\text{subject to } A_1 + A_2 \le 1, B_1 + B_2 \le 1, \text{all variables} \ge 0 \end{aligned} \tag{31}$$

The solutions are, after eliminating α

$$\begin{array}{|l|} \hline \text{Efficient allocations} \\ P = \{ (A_1, A_2, A_1, B_2) \in R_+^4 : A_1 + B_1 = 1, A_2 = 0, B_2 = 1 \} \\ \hline \text{Pareto frontier} \\ \text{PF} = \left\{ (v_A, v_B) \in R^2 : v_A + v_B = 2, 0 \le v_A \le 1 \right\} \\ \hline \text{Utility possibility set} \\ \text{UPS} = \left\{ (v_A, v_B) \in R^2 : v_A + v_B \le 2, v_B \le 2, v_A \le 1 \right\} \\ \hline \end{array} \tag{32}$$



EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$p_i =$ price of good $i, i = 1,2$

2. DEFINE CONSUMER INCOMES

$$M_A = p_1\alpha_1 + p_2\alpha_2, M_B = p_1(1-\alpha_1) + p_2(1-\alpha_2) \tag{33}$$

3. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

max $u_A = A_1$,subject to $p_1A_1 + p_2A_2 \le M_A$
max $u_B = B_1 + B_2$,subject to $p_1B_1 + p_2B_2 \le M_B$

The solutions are

$$(A_1, A_2) = \left(\frac{M_A}{p_1}, 0\right) \tag{34}$$

$$(B_1,B_2)=\begin{cases}\left(0,\dfrac{M_B}{p_2}\right) & \text{if } \dfrac{p_2}{p_1}<1\\ \left(\beta,\dfrac{M_B}{p_2}-\beta\right),0\le\beta\le\dfrac{M_B}{p_2} & \text{if } \dfrac{p_2}{p_1}=1\\ \left(\dfrac{M_B}{p_1},0\right) & \text{if } \dfrac{p_2}{p_1}>1\end{cases} \tag{35}$$

4. SOLVE THE EQUILIBRIUM CONDITIONS

$$1=A_1+B_1, 1=A_2+B_2 \tag{36}$$

$$\boxed{\begin{array}{l}\boxed{\text{equilibria}}\\ \boxed{\alpha_1+\alpha_2>1}\Rightarrow\dfrac{p_2}{p_1}=\dfrac{1-\alpha_1}{\alpha_2}\\ E(\omega_A)=\{(A_1,A_2,A_1,B_2)\in R_+^4: A_1=1,A_2=0,B_1=0,B_2=1\}\\ \boxed{\alpha_1+\alpha_2\le 1}\Rightarrow\dfrac{p_2}{p_1}=1\\ E(\omega_A)=\{(A_1,A_2,A_1,B_2)\in R_+^4: A_1=\alpha_1+\alpha_2,A_2=0,B_1=1-\alpha_1-\alpha_2,B_2=1\}\end{array}} \tag{37}$$

By (37),(32) $E(\omega_A)\subseteq P,\forall\omega_A$

$$\boxed{\begin{array}{l}E(\omega_A)\subseteq P,\forall\omega_A\\ \bigcup_{\omega_A}E(\omega_A)=P\end{array}} \tag{38}$$

QUESTION 5

THE ECONOMY

- Three goods, A, B, X, written in this order.
- Two consumers, 1 and 2
- Two firms, α and β

Consumer 1

- Consumption set R_+^3
- Endowment vector $\omega_1=[0,0,\gamma]$

- Profit shares $\theta_1 = [\theta_{1,\alpha}, \theta_{1,\beta}]$
- Utility function $U_1 = B_1$

Consumer 2

- Consumption set R^3_+
- Endowment vector $\omega_2 = [0, 0, \bar{X} - \gamma]$
- Profit shares $\theta_2 = [1-\theta_{1,\alpha}, 1-\theta_{1,\beta}]$
- Utility function $U_2 = A_2$

Firm α produces good A out of good X with technology described by the production function

$$A = \begin{cases} 0 & \text{if} \quad X_\alpha \le F \\ \sqrt{2(X_\alpha - F)} & \text{if} \quad X_\alpha \ge F \end{cases}, X_\alpha \ge 0$$

Firm β produces good B out of good X with technology described by the production function $B = X_\beta, X_\beta \ge 0$

Parameters: $\bar{X}, F, \gamma, \theta_1 = [\theta_{1,\alpha}, \theta_{1,\beta}]$.

Conditions on parameters: $0 \le \theta_{1,\alpha} \le 1, 0 \le \theta_{1,\beta} \le 1, 0 \le \gamma \le \bar{X}, 0 \le F < \bar{X}$

QUESTIONS

Answer the following questions for all allowed parameter values, keeping $\bar{X}, F$ fixed.

- Compute all efficient allocations P .Compute and draw the Pareto frontier and the utility possibility set.
- Compute all competitive equilibrium allocations $E(\gamma, \theta_1)$
- First welfare theorem: for which values of the parameters, if any, is it true that $E(\gamma, \theta_1) \subseteq P$?
- Compute the set of decentralizable efficient allocations $P \cap \left(\bigcup_\gamma \bigcup_{\theta_1} E(\gamma, \theta_1) \right)$. The union is taken over all allowed values of γ, θ_1
- second welfare theorem: for which values of the parameters $\bar{X}, F$, if any, is it true that all efficient allocations are decentralizable ,i.e. that $P \subseteq \bigcup_\gamma \bigcup_{\theta_1} E(\gamma, \theta_1)$?

ANSWERS

EFFICIENT ALLOCATIONS

The set of feasible allocations

$$
\begin{aligned}
&FA = FA_0 \cup FA_+, x = \left(B_1, A_2, A, X_\alpha, B, X_\beta\right),\\
&FA_0 = \left\{x \in R_+^6 : A = 0, A_2 \le A, B_1 \le B, X_\alpha + X_\beta \le \bar{X}, X_\beta \ge B, X_\alpha \le F\right\}\\
&FA_+ = \left\{x \in R_+^6 : A > 0, A_2 \le A, B_1 \le B, X_\alpha + X_\beta \le \bar{X}, X_\beta \ge B, X_\alpha \ge F + \frac{A^2}{2}\right\}
\end{aligned}
\tag{39}
$$

is not convex, hence we can only use the general method for computing efficient allocations, i.e. we solve for all values of the parameter θ_2 ,the following max problem

$$
\begin{aligned}
&\max U_1 = B_1\\
&\text{subject to } x \in FA, U_2 = A_2 \ge \theta_2
\end{aligned}
\tag{40}
$$

The solutions are

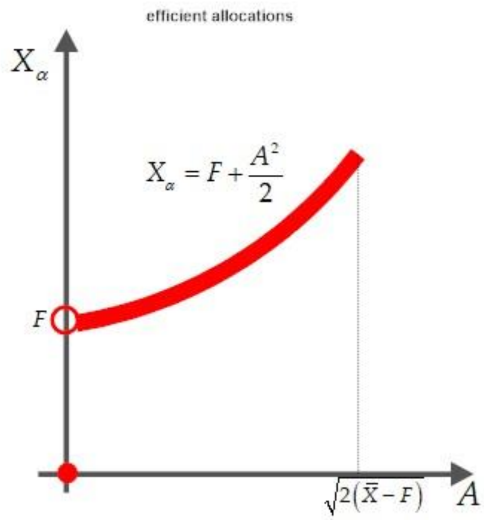
$$
\begin{aligned}
&\underline{\text{candidate efficient allocations}}\\
&x = \left(B_1, A_2, A, X_\alpha, B, X_\beta\right)\\
&\theta_2 \le 0 \Rightarrow x = \left(\bar{X}, 0, 0, 0, \bar{X}, \bar{X}\right)\\
&\theta_2 > \sqrt{2\left(\bar{X} - F\right)} \Rightarrow \text{no max exists}\\
&\theta_2 = \sqrt{2\left(\bar{X} - F\right)} \Rightarrow x = \left(0, \sqrt{2\left(\bar{X} - F\right)}, \sqrt{2\left(\bar{X} - F\right)}, \bar{X}, 0, 0\right)\\
&0 < \theta_2 < \sqrt{2\left(\bar{X} - F\right)} \Rightarrow x = \left(\bar{X} - F - \frac{\theta_2^2}{2}, \theta_2, \theta_2, F + \frac{\theta_2^2}{2}, \bar{X} - F - \frac{\theta_2^2}{2}, \bar{X} - F - \frac{\theta_2^2}{2}\right)
\end{aligned}
\tag{41}
$$

Since the solutions are essentially unique for each value of θ_2,we obtain from (41),after eliminating θ_2 ,

$$
\boxed{
\begin{aligned}
&\underline{\text{efficient allocations}}\\
&\boxed{P = P_0 \cup P_+}\\
&P_0 = \left\{\left(\bar{X}, 0, 0, 0, \bar{X}, \bar{X}\right)\right\}, x = \left(B_1, A_2, A, X_\alpha, B, X_\beta\right)\\
&P_+ = \left\{x \in R_+^6 : 0 < A \le \sqrt{2\left(\bar{X} - F\right)}, A_2 = A, B_1 = B = X_\beta = \bar{X} - F - \frac{A^2}{2}, X_\alpha = F + \frac{A^2}{2}\right\}
\end{aligned}
}
\tag{42}
$$

By (42) we obtain that the set of efficient allocations is isomorphic to a simpler set

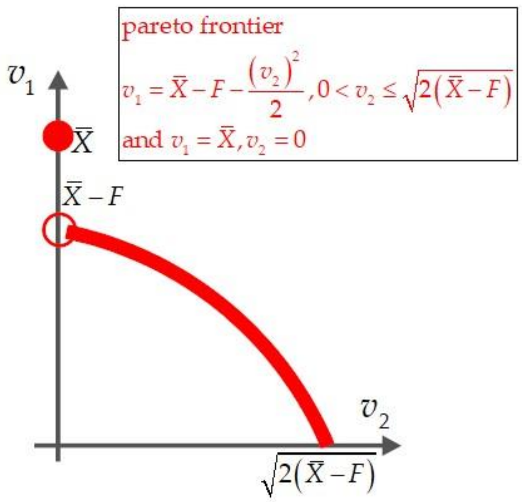
$$\boxed{P \approx PP_0 \cup PP_+}$$
$$PP_0 = \{(0,0)\}, y = (A, X_\alpha) \tag{43}$$
$$PP_+ = \{y \in R_+^2 : 0 < A \le \sqrt{2(\bar{X} - F)}, X_\alpha = F + \frac{A^2}{2}\}$$

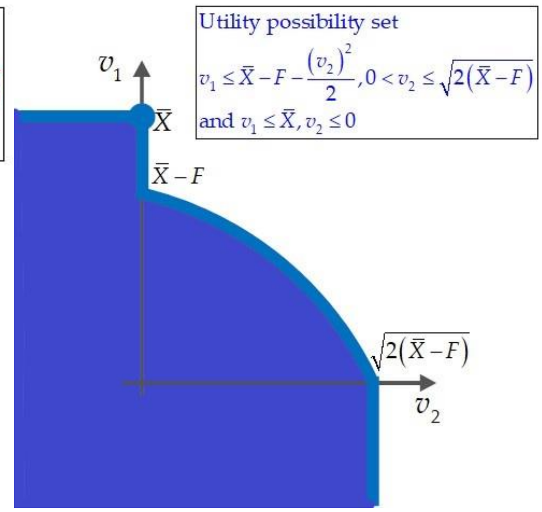


We obtain the pareto frontier by eliminating A from the system

$$v_1 = \bar{X} - F - \frac{A^2}{2}, v_2 = A, 0 \le A \le \sqrt{2(\bar{X} - F)}$$

$$\boxed{\begin{aligned} &\text{Pareto frontier} \\ &\text{PF} = \left\{(v_1, v_2)\right\} \in R^2 : v_1 = \bar{X} - F - \frac{(v_2)^2}{2}, 0 < v_2 \le \sqrt{2(\bar{X} - F)}\} \cup \{(\bar{X}, 0)\} \\ &\text{Utility possibility set} \\ &\text{UPS} = \text{UPS}_0 \cup \text{UPS}_+ \\ &\text{UPS}_0 = \left\{(v_1, v_2)\right\} \in R^2 : v_1 \le \bar{X} - F - \frac{(v_2)^2}{2}, 0 < v_2 \le \sqrt{2(\bar{X} - F)}\} \\ &\text{UPS}_+ = \left\{(v_1, v_2)\right\} \in R^2 : v_1 \le \bar{X}, v_2 \le 0\} \end{aligned}} \tag{44}$$





EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$p_A =$ price of good A, $p_B =$ price of good B, w=price of good X. Normalize $w = 1$

2. DEFINE CONSUMER INCOMES

$$M_1 = \gamma + \theta_{1,\alpha}\Pi_\alpha + \theta_{1,\beta}\Pi_\beta, M_2 = (\bar{X} - \gamma) + (1-\theta_{1,\alpha})\Pi_\alpha + (1-\theta_{1,\beta})\Pi_\beta \tag{45}$$

3. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

max $U_1 = B_1$, subject to $p_B B_1 \le M_1$, max $U_2 = A_2$, subject to $p_A A_2 \le M_2$

The solutions are

$$B_1 = \frac{M_1}{p_B}, A_2 = \frac{M_2}{p_A} \tag{46}$$

4. SOLVE THE OPTIMIZATION PROBLEMS OF FIRMS

$$\max \Pi_\alpha = \begin{cases} 0 & \text{if} \quad A = 0 \\ p_A A - \left(F + \dfrac{A^2}{2}\right) & \text{if} \quad A > 0 \end{cases}$$

$$\max \Pi_\beta = (p_B - 1) B$$

The solutions are

$$(B, X_\beta, \Pi_\beta) = \begin{cases} (0,0,0) & \text{if} & p_B < 1 \\ \{(\beta,\beta,0) : \beta \geq 0\} & \text{if} & p_B = 1 \\ (\infty,\infty,\infty) & \text{if} & p_B > 1 \end{cases} \qquad (47)$$

$$(A, X_\alpha, \Pi_\alpha) = \begin{cases} (0,0,0) & \text{if} \quad p_A < \sqrt{2F} \\ \{(0,0,0), (\sqrt{2F}, 2F, 0)\} & \text{if} \quad p_A = \sqrt{2F} \\ (p_A, F + \frac{1}{2} p_A{}^2, \frac{p_A{}^2}{2} - F) & \text{if} \quad p_A > \sqrt{2F} \end{cases} \qquad (48)$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$A_2 = A, B_1 = B, X_\alpha + X_\beta = \bar{X} \qquad (49)$$

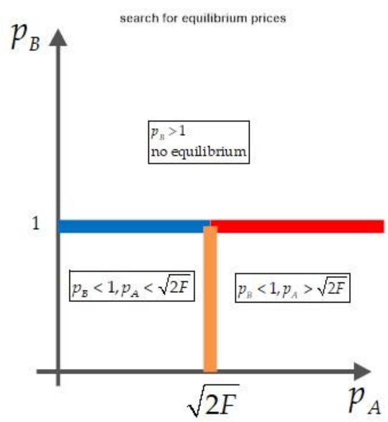
We start the search for equilibrium prices

Hypothesis (to be accepted or rejected): there is an equilibrium with $p_B > 1$

Consequences of the hypothesis

$X_\alpha + X_\beta = \bar{X}$ becomes $X_\alpha + \infty = \bar{X}$

Consistency check: The hypothesis is rejected

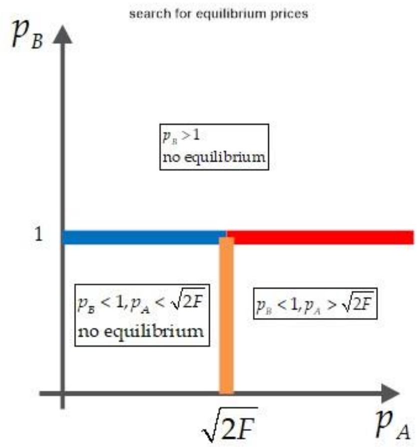


Hypothesis (to be accepted or rejected): there is an equilibrium with $p_B < 1, p_A < \sqrt{2F}$

Consequences of the hypothesis

$X_\alpha + X_\beta = \bar{X}$ becomes $0 + 0 = \bar{X}$

Consistency check: The hypothesis is rejected



Hypothesis (to be accepted or rejected): there is an equilibrium with $p_B < 1, p_A \geq \sqrt{2F}$

Consequences of the hypothesis

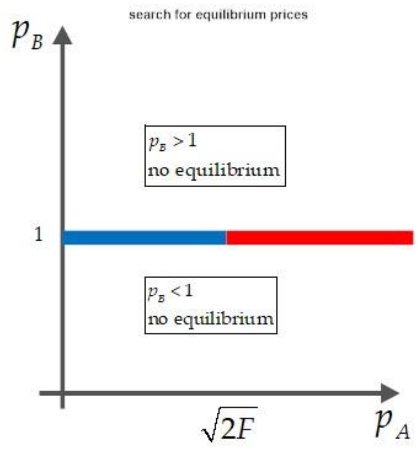
$A_2 = A, B_1 = B, X_\alpha + X_\beta = \bar{X}$ become

$$\frac{M_2}{p_A} = p_A, \frac{M_1}{p_B} = 0, F + \frac{1}{2} p_A{}^2 + 0 = \bar{X},$$

$$\Pi_\alpha = \frac{p_A{}^2}{2} - F, \Pi_\beta = 0$$

$$M_1 = \gamma + \theta_{1,\alpha} \Pi_\alpha, M_2 = \left(\bar{X} - \gamma\right) + \left(1 - \theta_{1,\alpha}\right) \Pi_\alpha$$

Consistency check: The hypothesis is rejected



Hypothesis (to be accepted or rejected): there is an equilibrium with $p_B = 1, p_A < \sqrt{2F}$

Consequences of the hypothesis

$A_2 = A, B_1 = B, X_\alpha + X_\beta = \bar{X}$ become

$$\frac{M_2}{p_A} = 0, \frac{M_1}{p_B} = B = X_\beta = \bar{X},$$

$$\Pi_\alpha = 0, \Pi_\beta = 0$$

$$M_1 = \gamma, M_2 = \left(\bar{X} - \gamma\right)$$

Consistency check: The hypothesis is accepted iff $\gamma = \bar{X}$

equilibria

$$\boxed{\gamma = \bar{X}} \Rightarrow 0 < p_A < \sqrt{2F}, p_B = 1, A = A_2 = X_\beta = 0, B_1 = B = X_\beta = \bar{X} \qquad (50)$$

$$\boxed{\gamma \neq \bar{X}} \Rightarrow ?$$

Hypothesis (to be accepted or rejected): there is an equilibrium with $p_B = 1, p_A \geq \sqrt{2F}$

Consequences of the hypothesis

$A_2 = A, B_1 = B, X_\alpha + X_\beta = \bar{X}$ become

$$\frac{M_2}{p_A} = p_A, \frac{M_1}{p_B} = B = X_\beta, F + \frac{1}{2} p_A^{\ 2} + X_\beta = \bar{X},$$

$$\Pi_\alpha = \frac{p_A^{\ 2}}{2} - F, \Pi_\beta = 0$$

$$M_1 = \gamma + \theta_{1,\alpha}\left(\frac{p_A^{\ 2}}{2} - F\right), M_2 = \left(\bar{X} - \gamma\right) + \left(1 - \theta_{1,\alpha}\right)\left(\frac{p_A^{\ 2}}{2} - F\right)$$

The solution is

$$\begin{aligned}
p_A &= \sqrt{2}\sqrt{\frac{\bar{X} - \gamma - F\left(1 - \theta_{1,\alpha}\right)}{1 + \theta_{1,\alpha}}}, p_B = 1, \Pi_\alpha = \frac{\bar{X} - \gamma - 2F}{1 + \theta_{1,\alpha}} \\
B_1 &= B = X_\beta = \frac{\gamma + \left(\bar{X} - 2F\right)\theta_{1,\alpha}}{1 + \theta_{1,\alpha}}, \Pi_\beta = 0 \\
A &= \sqrt{2}\sqrt{\frac{\bar{X} - \gamma - F\left(1 - \theta_{1,\alpha}\right)}{1 + \theta_{1,\alpha}}}, X_\alpha = \frac{\bar{X} - \gamma + 2\theta_{1,\alpha}F}{1 + \theta_{1,\alpha}} \\
M_1 &= \frac{\gamma + \theta_{1,\alpha}(\bar{X} - 2F)}{1 + \theta_{1,\alpha}}, M_2 = \frac{2\left(\bar{X} - \gamma - F\left(1 - \theta_{1,\alpha}\right)\right)}{1 + \theta_{1,\alpha}}
\end{aligned} \tag{51}$$

Consistency check: we substitute (51) into the system of inequalities

$\left\{B \geq 0, \Pi_\alpha \geq 0, p_A \geq \sqrt{2F}\right\}$ yields $0 \leq \gamma \leq \bar{X} - 2F, 0 \leq F \leq \frac{\bar{X}}{2}$

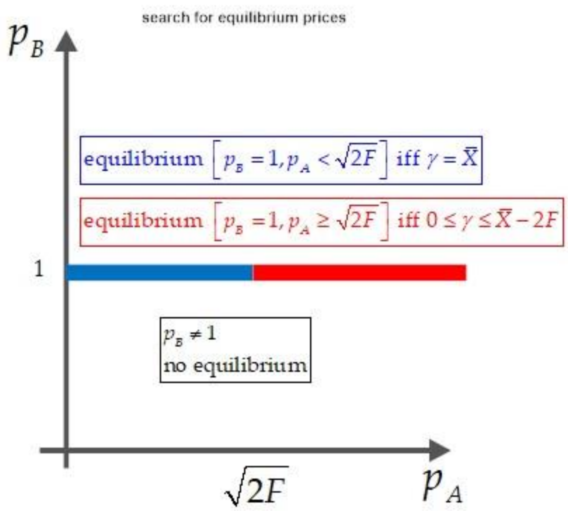
equilibria

$$\boxed{\gamma=\bar{X}} \Rightarrow 0<p_A<\sqrt{2F}, p_B=1, A=A_2=X_\beta=0, B_1=B=X_\beta=\bar{X}$$

$$\boxed{0\le\gamma\le\bar{X}-2F, 0\le F\le\frac{\bar{X}}{2}} \Rightarrow p_A=\sqrt{2}\sqrt{\frac{\bar{X}-\gamma-F\left(1-\theta_{1,\alpha}\right)}{1+\theta_{1,\alpha}}}, p_B=1, \Pi_\alpha=\frac{\bar{X}-\gamma-2F}{1+\theta_{1,\alpha}} \quad (52)$$

$$B_1=B=X_\beta=\frac{\gamma+\left(\bar{X}-2F\right)\theta_{1,\alpha}}{1+\theta_{1,\alpha}}, A=\sqrt{2}\sqrt{\frac{\bar{X}-\gamma-F\left(1-\theta_{1,\alpha}\right)}{1+\theta_{1,\alpha}}}, X_\alpha=\frac{\bar{X}-\gamma+2\theta_{1,\alpha}F}{1+\theta_{1,\alpha}}$$

$$\boxed{\left[\frac{\bar{X}}{2}<F<\bar{X}\right] \text{ or } \left[0\le F\le\frac{\bar{X}}{2} \text{ and } \bar{X}-2F<\gamma<\bar{X}\right]} \Rightarrow ?$$



Having exhausted the $\left(p_A, p_B\right)$ search space, we conclude from (52) that

equilibria

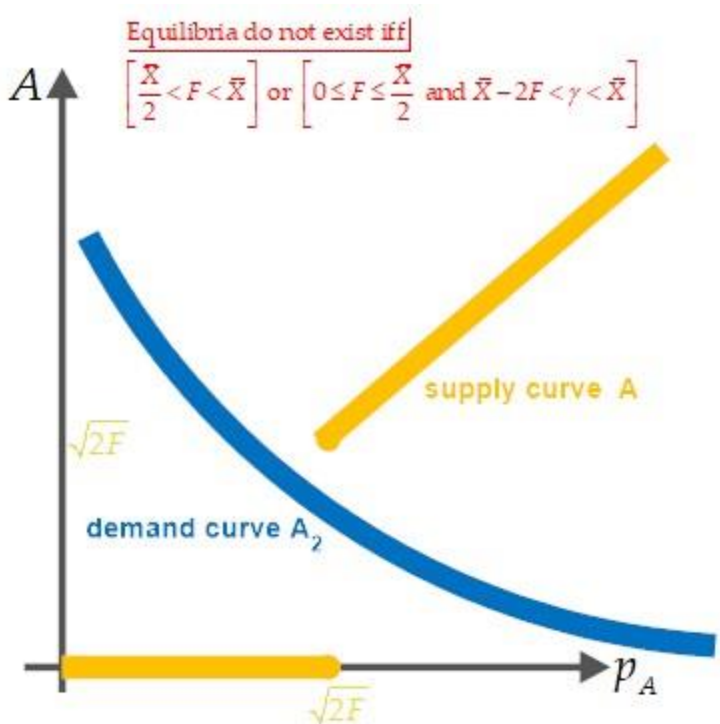
$$\boxed{\gamma=\bar{X}} \Rightarrow 0<p_A<\sqrt{2F}, p_B=1, x=\left(B_1, A_2, A, X_\alpha, B, X_\beta\right)$$

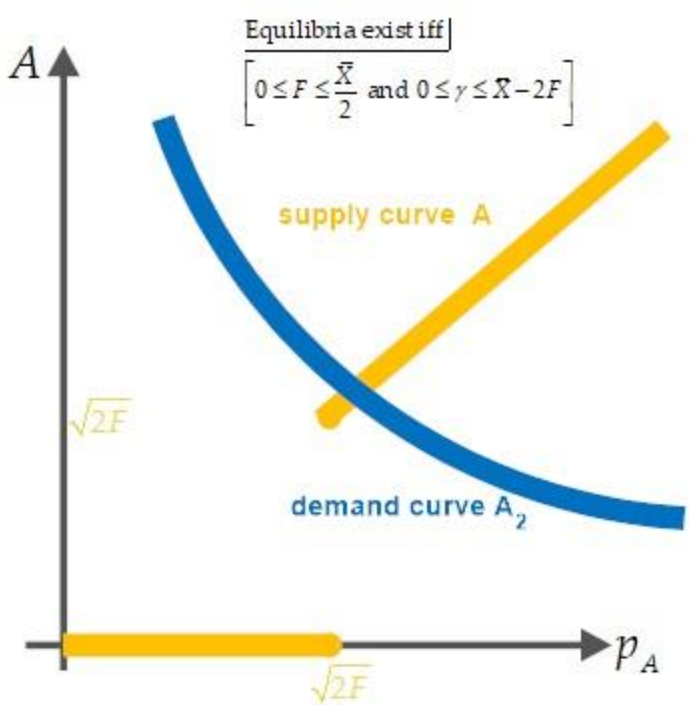
$$E\left(\gamma,\theta_1\right)=\left\{x\in R_+^6 : A_2=A=X_\beta=0, B_1=B=X_\beta=\bar{X}\right\}$$

$$\boxed{0\le\gamma\le\bar{X}-2F, 0\le F\le\frac{\bar{X}}{2}} \Rightarrow p_A=\sqrt{2}\sqrt{\frac{\bar{X}-\gamma-F\left(1-\theta_{1,\alpha}\right)}{1+\theta_{1,\alpha}}}, p_B=1, \Pi_\alpha=\frac{\bar{X}-\gamma-2F}{1+\theta_{1,\alpha}} \quad (53)$$

$$E\left(\gamma,\theta_1\right)=\left\{x\in R_+^6 : B_1=B=X_\beta=\bar{X}-F-\frac{A^2}{2}, A_2=A=p_A, X_\alpha=F+\frac{A^2}{2}\right\}$$

$$\boxed{\left[\frac{\bar{X}}{2}<F<\bar{X}, 0\le\gamma<\bar{X}\right] \text{ or } \left[0\le F\le\frac{\bar{X}}{2} \text{ and } \bar{X}-2F<\gamma<\bar{X}\right]} \Rightarrow E\left(\gamma,\theta_1\right)=\varnothing$$





Let $E = \bigcup_{\gamma} \bigcup_{\theta_1} E(\gamma, \theta_1)$ .By (53),(42) $E(\gamma, \theta_1) \subseteq P, \forall \gamma, \forall \theta_1$,hence

$$\boxed{E \subseteq P} \tag{54}$$

and therefore

$$\boxed{\begin{array}{l}\text{Decentralizable efficient allocations}\\ \boxed{P\bigcap E=E}\end{array}} \tag{55}$$

By (53)

$$\boxed{\frac{\bar{X}}{2}<F<\bar{X}\Rightarrow E\approx\{(0,0)\}} \tag{56}$$

$$0\le F\le\frac{\bar{X}}{2}\Rightarrow$$
$$E=\bigcup_{0\le\gamma\le\bar{X}-2F}\bigcup_{\theta_1}\left\{x\in R_+^6:B_1=B=X_\beta=\bar{X}-F-\frac{A^2}{2},A_2=A=\sqrt{2}\sqrt{\frac{\bar{X}-\gamma-F\left(1-\theta_{1,\alpha}\right)}{1+\theta_{1,\alpha}}},X_\alpha=F+\frac{A^2}{2}\right\} \tag{57}$$

By (57) $\partial A/\partial\theta_{1,\alpha}<0,\partial A/\partial\gamma<0$,hence the equilibrium values of A are $\sqrt{2F}\le A\le\sqrt{2\left(\bar{X}-F\right)}$ .Finally, for $y=\left(X_\alpha,A\right)$ we obtain

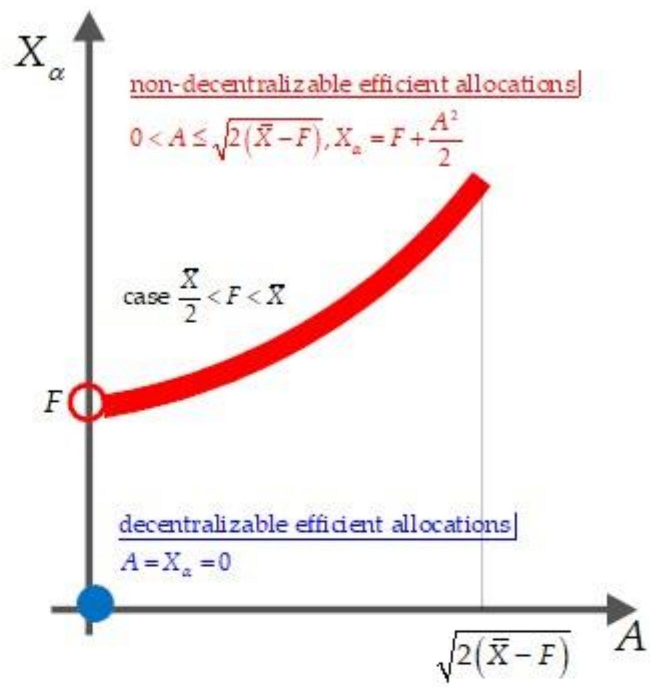
$$\boxed{\begin{array}{l}\boxed{\text{equilibrium allocations } E}\\ \boxed{E\approx\begin{cases}PP_0 & \text{if}\quad \frac{\bar{X}}{2}<F<\bar{X}\\ PP_0\bigcup EE_+ & \text{if}\quad 0\le F\le\frac{\bar{X}}{2}\end{cases}}\\ PP_0=\{(0,0)\}\\ EE_+=\{y\in R_+^2:\sqrt{2F}\le A\le\sqrt{2\left(\bar{X}-F\right)},X_\alpha=F+\frac{A^2}{2}\}\end{array}} \tag{58}$$

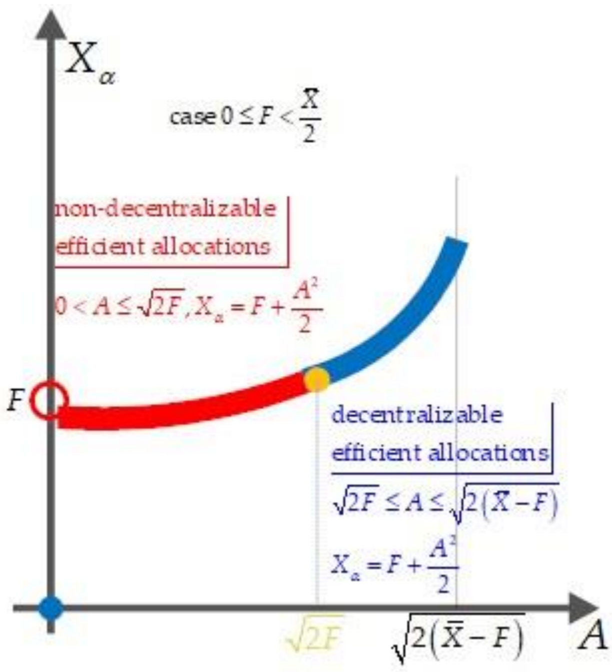
By (58),(43) we have, for $y=\left(X_\alpha,A\right)$

$$\boxed{\begin{array}{l}\underline{\text{non-decentralizable efficient allocations}} \\ P-E=\begin{cases}\{y\in R_+^2: 0<A\le\sqrt{2\left(\bar{X}-F\right)}, X_\alpha=F+\frac{A^2}{2}\} & \text{if } \frac{\bar{X}}{2}<F<\bar{X} \\ \{y\in R_+^2: 0<A<\sqrt{2F}, X_\alpha=F+\frac{A^2}{2}\} & \text{if } 0\le F\le\frac{\bar{X}}{2}\end{cases}\end{array}} \quad (59)$$

$P-E$ measures the extent of the tradeoff between equity and efficiency. Since $P\subseteq E$ iff $P - E=\varnothing$,we obtain by (59) that the extent of the tradeoff increases in F ,and that it disappears iff $F=0$

$$\boxed{P\subseteq E \text{ iff } P - E=\varnothing \text{ iff } F=0} \quad (60)$$




X_α
case $0 \le F < \frac{\bar{X}}{2}$
non-decentralizable
efficient allocations
$0 < A \le \sqrt{2F}, X_\alpha = F + \frac{A^2}{2}$
F
decentralizable
efficient allocations
$\sqrt{2F} \le A \le \sqrt{2(\bar{X} - F)}$
$X_\alpha = F + \frac{A^2}{2}$
$\sqrt{2F}$
$\sqrt{2(\bar{X} - F)}$
A