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An aggregate excess demand function is any function that can be written as a finite sum of individual excess demand functions $J_1(p) + \dots + J_n(p)$, with each J_i satisfying H-W-B-SARP

$$\boxed{\text{Properties H-W-B are hereditary, ie they are satisfied by any aggregate excess demand function}}$$

Proof: Let $Z = J_1 + \dots + J_n$, with each $J_i : R^L_{++} \to R^L$ satisfying H-W-B

Step 1: Z is homogeneous of degree 0

$$Z(\theta p) = J_1(\theta p) + \quad + J_n(\theta p) = J_1(p) + \dots + J_n(p) = Z(p), \quad \forall \theta > 0$$

Step 2: Z satisfies Walras law

$$pZ(p) = pJ_1(p) + \quad + pJ_n(p) = 0 + \dots + 0 = 0$$

Step 3: Z is bounded from below

Let $J^i(p) + b^i \gg 0 \quad \forall p \gg 0 \ \forall i$. Then

$$Z(p) = \sum_i J^i(p), \quad \text{hence} \quad Z(p) + \sum b^i = \sum_i (J^i(p) + b^i) > 0 \ \forall p.$$

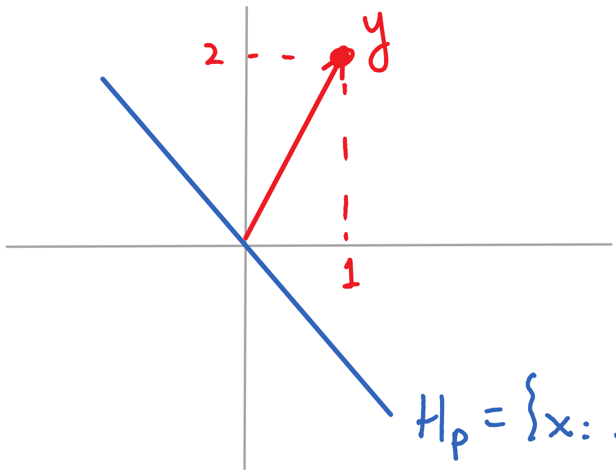
SARP is not hereditary. This is the content of the SONNENSCHEIN-MANTEL-DEBREU THEOREM

Any function $Z: R^L_H \to R^L$ that satisfies H-W-B is an aggregate excess demand function

p, y are nonzero vectors in R^L, $py \neq 0$

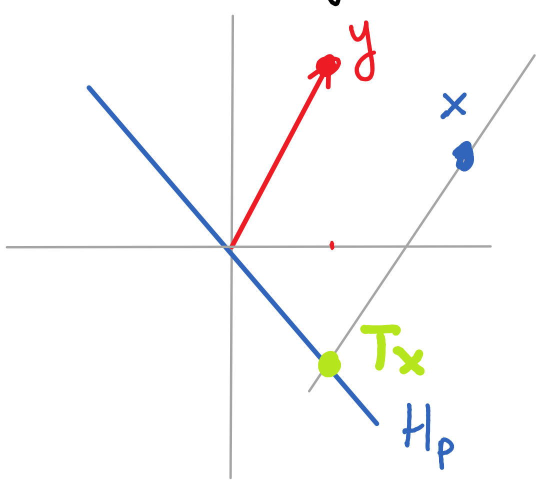
Example 1: $p = (1,1)$, $y = (2,1)$

The hyperplane defined by p $\quad H_p = \{x \in R^L : px = 0\}$



$H_p = \{x : px = 0\} = \{x : x_1 + x_2 = 0\}$

Oblique projection of a point x on the hyperplane H_p along the direction y



Draw a line through x, parallel to y, and intersect it with H_p

Tx is the oblique projection of x on H_p along y

Sometimes we write $T = T_{p,y}$

Projection on Hp along y

Theorem: $\boxed{Tx = \left(I - \frac{1}{py} y p^T\right) x}$

Proof: $\{x + \lambda y : \lambda \in R\}$ is the equation of the line through x parallel to y. Hence there exists some θ such that $Tx = x + \theta y$, $p(Tx) = 0$.

Then $0 = p(x + \theta y) = px + \theta py$ ie

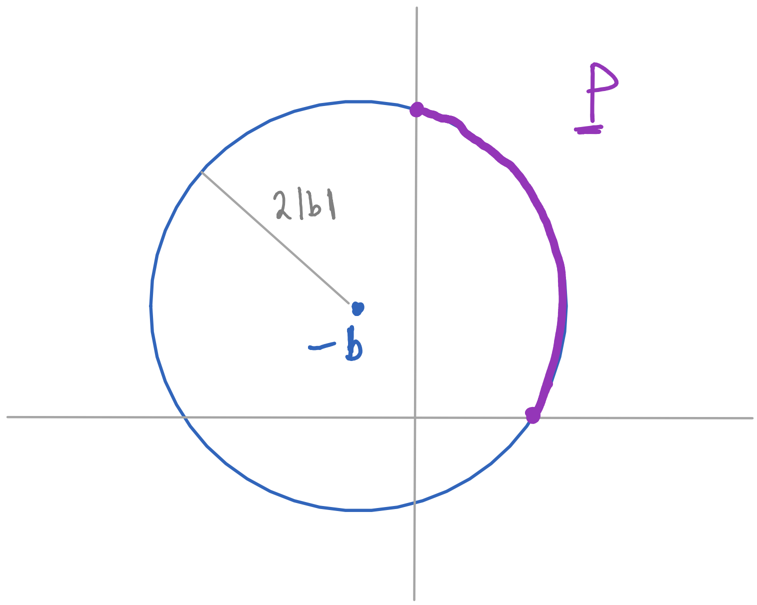
$$\theta = -\frac{px}{py}, \quad Tx = x - \frac{px}{py} y, \text{ ie}$$

$$Tx = \left(I - \frac{1}{py} y p^T\right) x$$

Restriction

$b \gg 0$ in R^L, $\mathcal{P} = \{ x \in R^L_{++} : |x+b| = 2|b| \}$

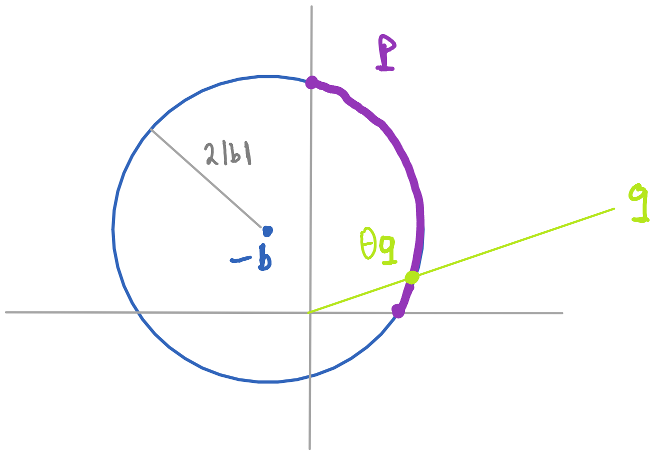
$\mathcal{P}$ is the sphere with center $-b$ and radius $2|b|$, restricted to the positive orthant



PROPERTIES OF $\mathcal{P}$

(1) $b \in \mathcal{P}$

(2) For each $q \gg 0$ there is a unique number $\theta > 0$ such that $\theta q \in \mathcal{P}$



Proof: The equation $|\theta q + b| = 2|b|$ squared yields

$$\theta^2 |q|^2 + 2\theta bq + |b|^2 = 4|b|^2 \quad \text{ie}$$

$$|q|^2 \theta^2 + 2bq\theta - 3|b|^2 = 0$$

The discriminant $\Delta = 4(bq)^2 + 12|b|^2|q|^2 > 0$, hence there are two real roots. The product of the roots is

$-3\dfrac{|b|^2}{|q|^2} < 0$, hence there is exactly one positive root,

namely $$\theta = \frac{-bq + \sqrt{(bq)^2 + 3|b|^2|q|^2}}{|q|^2}$$

Reduction to P

(3) Each $q >> 0$ can be written as a positive multiple of some $p \in P$,

$$q = \lambda(q) \cdot p, \quad p \in P$$

and λ is homogeneous of degree 1

$$\lambda(\gamma q) = \gamma\, \lambda(q), \quad \gamma > 0$$

and λ equals one on $\mathcal{P}$

$$\lambda(p) = 1 \quad \forall p \in \mathcal{P}$$

Proof: Let $q \gg 0$; By (2), there exists a unique $\theta > 0$ such that $\theta q \in \mathcal{P}$. Then $q = \frac{1}{\theta}(\theta q)$, i.e. $\lambda = \frac{1}{\theta}$, $x = \theta q$. By (2) again

$$\lambda(q) = \frac{1}{\theta} = \frac{|q|^2}{-bq + \sqrt{(bq)^2 + 3|b|^2|q|^2}}$$

. Clearly

$$\lambda(\gamma q) = \gamma\, \lambda(q), \quad \gamma > 0.$$

Finall if $q \in \mathcal{P}$, ie if $|q + b| = 2|b|$, then $\lambda = \lambda(q) = 1/\theta$, where $|\theta q + b| = 2|b|$. By uniqueness, $\theta = 1$. Hence $\lambda(q) = 1$

(4) For any two $p, q \in \mathcal{P}$ with $p \neq q$

$$p(q+b) < q(q+b)$$

Proof. Consider the maximization problem

$$\text{MAX} \quad f(p) = p(q+b)$$

subject to $|p+b|^2 \leq 4|b|^2$, $p \geqslant 0$ $\qquad$ (M)

Variables: p. Parameters: q, b

Conditions on parameters: $b >> 0$, $|q+b| = 2|b|$, $q >> 0$.

We will show that the unique global maximum is $p = q$. This suffices to prove the inequality $p(q+b) < q(q+b)$, $\forall p \in \mathcal{P}$, $\forall q \in \mathcal{P}$

Step 1: (M) has a global maximum, by Weierstrass.

The Lagrangian of the problem is

$$L = \lambda_0 \, p(q+b) + \lambda_1 \left(4|b|^2 - |p+b|^2 \right)$$

Step 2: If x is a global maximum, then $|x+b| = 2|b|$. In particular, $x \neq 0$.

If $|x+b|^2 < 4|b|^2$, $x+\epsilon$ is feasible and better.

Step 3 If x is a global maximum, then there exist $\lambda_0 \in \{0,1\}$ $\lambda_1 \geqslant 0$, not all zero, such that

$$\left.\frac{\partial L}{\partial p}\right|_{p=x} = \lambda_0 (q+b) - 2\lambda_1 (x+b) \leq 0$$

$$x \left.\frac{\partial L}{\partial p}\right|_{p=x} = x\left(\lambda_0 (q+b) - 2\lambda_1 (x+b) \right) = 0$$

FRITZ JOHN NECESSARY CONDITIONS

$\partial p\big|_{p=x}$

$|x+b| = 2|b|$

CONDITIONS

Step 4 : $\lambda_0 = 1$, for if $\lambda_0 = 0$, then by step 3 $\lambda_1 > 0$, hence $\left.\frac{\partial L}{\partial p}\right|_{p=x} < 0$, hence $x = 0$, a contradiction

Step 5 $\lambda_1 > 0$, for otherwise $q + b \leq 0$, a contradiction

Step 6 : (M) satisfies the prerequisites of the Arrow-Enthoven theorem, because the objective function is linear and the feasible set is (strictly) convex. To see this let $g(p) = 4|b|^2 - |p+b|^2$. Then $S = \{p \geq 0,\ g(p) \geq 0\}$. Since

$g(p_1, \ldots, p_n) = 4|b|^2 - \sum_{i=1}^{L}(p_i + b_i)^2$, its Hessian is

$H = -2I$, which is negative definite. Hence g is strictly concave, making S strictly convex.

Step 6 If $x \gg 0$ then $x = q$, because $x \gg 0$ implies $\left.\frac{\partial L}{\partial p}\right|_{p=x} = 0$, ie $q + b = 2\lambda_1(x+b)$, ie

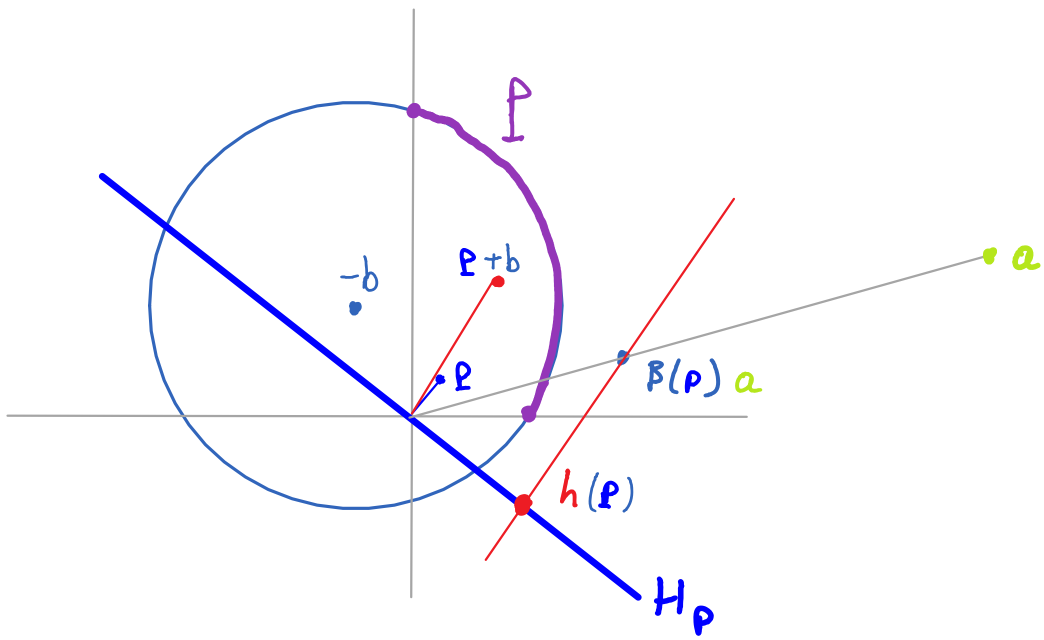
$|q+b| = 2\lambda_1|x+b| = 4\lambda_1|b|$

ie $\lambda_1 = \dfrac{|a+b|}{4|b|}$, ie $x+b = \underbrace{\left(\dfrac{2|b|}{|a+b|}\right)}_{1}(a+b)$, ie

$x+b = a+b$ ie $x = a$

For each $a \geqslant 0$, $a \neq 0$, and each $\beta : \mathcal{P} \to R_{++}$ we define a function $h : \mathcal{P} \longrightarrow R^L$ by

$$\boxed{h(p) = T_{p,p+b}(\beta(p)\,a) = \left(I - \frac{1}{p^T(p+b)}(p+b)\,p^T\right)\beta(p)\,a}$$



h satisfies Walras' law on $\mathcal{P}$, ie $p\,h(p) = 0 \quad \forall p \in \mathcal{P}$, because

$$p^T h(p) = p^T\left(I - \frac{1}{p^T(p+b)}(p+b)\,p^T\right)\beta(p)\,a =$$

$$\left(p^T - \frac{\cancel{p^T(p+b)}}{\cancel{p^T(p+b)}}\,p^T\right)\beta(p)\,a = \left(p^T - p^T\right)\beta(p)\,a = 0.$$

h is injective, ie $h(p) = h(q)$ implies $p = q$.

Proof: Suppose $h(p) = h(q)$. Show that $p = q$.

$$0 = p^T h(p) = p^T h(q) = \beta(q) \left(p^T - \frac{p^T(q+b)}{q^T(q+b)} q^T \right) a, \text{ Hence}$$

$$p^T a = \left(\frac{p^T(q+b)}{q^T(q+b)} \right) q^T a. \quad \text{Similarly}$$

$$0 = q^T h(q) = q^T h(p) \quad \text{implies}$$

$$q^T a = \left(\frac{q^T(p+b)}{p^T(p+b)} \right) p^T a. \quad \text{Hence}$$

$$\underbrace{\cancel{p^T a}}_{1} = \left(\frac{p^T(q+b)}{q^T(q+b)} \right) \cdot \left(\frac{q^T(p+b)}{p^T(p+b)} \right) \cdot \cancel{p^T a}$$

If $p \neq q$, then the terms in parentheses are strictly less than 1, a contradiction. Hence $p = q$

For any two $p \neq q$ in $\mathcal{P}$, $p h(q) \leq 0$ implies $pa < qa$

Proof: $p^T h(q) \leq 0$ implies

$$pa = p^T a \leq \left(\frac{p^T(q+b)}{p^T(p+b)} \right) q^T a < q^T a = qa.$$

h satisfies SARP on $\mathcal{P}$

Proof Let $p = p^1 \to p^2 \to \cdots \to p^{N-1} \to p^N = p$ on $\mathcal{P}$. Then

$p = p^1 \neq p^2 \neq \dots \neq p^{N-1} \neq p^N = p$ and $p^i h(p^{i+1}) \leq 0 \;\; \forall i$, ie

$p^i a < p^{i+1} a \quad \forall i$. Hence

$p a = p^1 a < p^2 a < \dots < p^{N-1} a < p^N a = p a$, contradiction

MAIN LEMMA

h extends to a function $\hat{h} : \mathbb{R}^L_{++} \to \mathbb{R}^L$ that satisfies H-W-SARP and is proportionally injective

$$\begin{tikzcd} \mathcal{P} \arrow[r, "inc"] \arrow[dr, "h"'] & \mathbb{R}^L_{++} \arrow[d, "\hat{h}"] \\ & \mathbb{R}^L \end{tikzcd} \qquad \hat{h}(p) = h(p) \quad \forall p \in \mathcal{P}$$

Proof:

Step 1: Extend h to $\mathbb{R}^L_{++}$, by Reduction to P

$$\hat{h}(x) = h\left(\frac{x}{\partial(x)}\right), \qquad x \gg 0$$

Step 2: $\hat{h}$ is homogeneous of degree 0, because ∂ is homogeneous of degree 1, hence $\hat{h}(\theta x) = h\left(\frac{\theta x}{\partial(\theta x)}\right) = h\left(\frac{\theta x}{\theta \partial(x)}\right) =$

$$= h\left(\frac{x}{\partial(x)}\right) = \hat{h}(x), \quad \forall \theta > 0$$

Step 3: $\hat{h}$ satisfies Walras' law

Let $\theta = \frac{1}{\partial(p)}$. Then

$$p\,\hat{h}(p) = p\,h(\theta p) = \frac{1}{\theta}(\theta p)\,h(\theta p) = 0.$$

Step 4: $\hat{h}$ is proportionally 1-1, ie $\hat{h}(p) = \hat{h}(q)$ iff p and q are proportional

If p, q are proportional then $q = \theta p$ for some $\theta > 0$.

Hence $\hat{h}(q) = \hat{h}(\theta p) = \hat{h}(p)$.

If $\hat{h}(p) = \hat{h}(q)$ then $h(p/\partial(p)) = h(q/\partial(q))$. Since h is injective, $p/\partial(p) = q/\partial(q)$ ie $q = \left(\frac{\partial(q)}{\partial(p)}\right)p$, ie p, q are proportional

Step 5: $\hat{h}$ satisfies SARP on R^L_{++}

Suppose not. Then the relation induced by $\hat{h}$ on prices has a cycle, say $p = p^1 \to p^2 \to \dots \to p^{N-1} \to p^N = p$

Then $\hat{h}(p^i) \neq \hat{h}(p^{i+1})$, $p^i \hat{h}(p^{i+1}) \leq 0$ $\forall i = 1, \dots, N-1$

Since $\hat{h}$ is proportionally 1-1, we conclude that

p^i is not proportional to p^{i+1} $\forall i$

$p^i \hat{h}(p^{i+1}) \leq 0$ $\forall i$

Let $\theta_i = 1/\partial(p^i)$. Then by the definition of $\hat{h}$

p^i is not proportional to p^{i+1} (1)

$p^i h(\theta_{i+1} p^{i+1}) \leq 0$ $\forall i$ (2)

$\theta_i p^i \in \underline{P}$ (3)

$$\theta_i p^i \in \underline{P} \qquad (3)$$

By (2), $\left(\theta_i p^i\right) h(\theta_{i+1} p^{i+1}) \leq 0 \quad \forall i.$ $\qquad (4)$

$$h(\theta_i p^i) \neq h(\theta_{i+1} p^{i+1}) \qquad (5)$$

By (3) (4) (5) we conclude that the relation induced by h on $\underline{P}$ satisfies, $\theta_1 p = \theta_1 p^1 \longrightarrow \theta_2 p^2 \longrightarrow \cdots \longrightarrow \theta_N p^N = \theta_N p$.

At the same time, θ_1, θ_N are defined by $\theta_1 p \in \underline{P}$, $\theta_N p \in \underline{P}$, hence by uniqueness, $\theta_1 = \theta_N$. Hence h induces a cycle on $\underline{P}$, contradicting the fact that h satisfies SARP on $\underline{P}$

SONNENSCHEIN – MANTEL – DEBREU

SMD THEOREM: $Z: R^L_{++} \to R^L$ is an aggregate excess demand function iff it satisfies H–W–B

Proof:

Necessity: If Z is an aggregate excess demand function then it inherits properties H–W–B from the individual excess demand functions $J_1, \dots, J_n$ that satisfy $Z = \sum J_i$

Sufficiency: Let Z satisfy H–W–B. We will construct functions $J_1, \dots, J_L$ that satisfy H–W–B–SARP and $Z = \sum J_i$. We use the following notation

$$e^i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in R^L \quad \text{has } 1 \text{ in position } i, \ 0 \text{ elsewhere}$$

$Z(p) + b >> 0 \quad \forall p >> 0$, and $b >> 0$ (Z satisfies B)

$\beta_i(p) = Z_i(p) + p_i + b_i$

$h^i(p) = T_{p, p+b}(\beta_i(p) e^i)$, $p \in P$

$\hat{h}^i$ = extension of h^i to R^L_{++}

Recall that each $\hat{h}^i$ satisfies H-W-SARP and is proportionally injective

Step 1: $z(p) = h^1(p) + \ldots + h^L(p) \quad \forall p \in P$

$$\sum_{i=1}^{L} h^i(p) = \sum_{i=1}^{L} T_{p,p+b}(\beta_i(p) e^i) = T_{p,p+b}\left(\sum_{i=1}^{L} \beta_i(p) e^i\right)$$

$$= T_{p,p+b}(z(p) + p + b) = \left(I - \frac{1}{p^T(p+b)}(p+b)p^T\right)(z(p)+p+b)$$

$$= z(p) + p + b - \frac{1}{p^T(p+b)}(p+b)p^T(z(p)+p+b)$$

$$= z(p) + p + b - \frac{1}{\cancel{p^T(p+b)}}\left[(p+b)\underbrace{\cancel{p^T z(p)}}^{0} + (p+b)\cancel{p^T(p+b)}\right]$$

$$= z(p) + p + b - (p+b) = z(p)$$

Step 2: $z(p) = \hat{h}^1(p) + \ldots + \hat{h}^L(p) \quad \forall p \gg 0$

$$\sum_{i=1}^{L} \hat{h}^i(p) = \sum_{i=1}^{L} h^i\left(\frac{p}{\lambda(p)}\right) = z\left(\frac{p}{\lambda(p)}\right) = z(p)$$

Step 3: $\exists\ \omega \gg 0$ such that $\forall i = 1, \ldots, L \quad \forall p \in P$

$$h^i(p) + w \gg 0$$

For each $p \in P$ and each $i = 1, \ldots, L$

$$h^i(p) = \left(I - \frac{1}{p^T(p+b)}(p+b)p^T\right)\beta_i(p)e^i$$

$$= \beta_i(p)e^i - \frac{p_i\beta_i(p)}{p^T(p+b)}(p+b)$$

$$\geq -\frac{p_i\beta_i(p)}{p^T(p+b)}(p+b)$$

$$= -\frac{p_i(z_i(p) + p_i + b_i)}{p^T(p+b)}(p+b)$$

$$= -\left(\frac{p_i(z_i(p)+b_i) + p_i^2}{p^T(p+b)}\right)(p+b) \quad .$$

Note that $p_i(z_i(p)+b_i) + p_i^2 > 0 \quad \forall\, p \gg 0$ Hence

$$\sum_{i=1}^{L}\left(p_i(z_i(p)+b_i)+p_i^2\right) > p_i(z_i(p)+b_i)+p_i^2 \quad \forall i \quad \text{ie}$$

$$\sum_{i=1}^{L} p_i(p_i+b_i) > p_i(z_i(p)+b_i)+p_i^2 \quad \forall i \quad \text{ie}$$

$$p^T(p+b) > p_i(z_i(p)+b_i+p_i) \quad \forall i \quad , \quad \text{Hence}$$

$$\frac{p_i(z_i(p)+b_i+p_i)}{p^T(p+b)} < 1 \quad , \quad \text{hence}$$

$h^i(p) \geq -(p+b) \qquad \forall p \in \mathcal{P} \quad \forall i \qquad$ i.e.

$h^i(p) + p+b \geq 0 \qquad \forall i \quad \forall p \in \mathcal{P}.$

Since $p \in \mathcal{P}$, we have $|p+b| = 2|b|$, $p >> 0$.

Let $w_i = 1 + 2|b|$ $\qquad$ Then $p+b << w \quad \forall p \in \mathcal{P}$

and $\quad h^i(p) + w >> h^i(p) + p+b \geq 0 \quad \forall i \quad \forall p \in \mathcal{P}$

Step 4: $\hat{h}^i$ is bounded below on R^L_{++}

$$\hat{h}^i(p) + w = h^i\left(\frac{p}{\lambda(p)}\right) + w >> 0 \qquad \forall p >> 0$$