Players: A, B

$\boxed{\text{Economy}}$ $\langle X, U_A, U_B \rangle$

X = set of outcomes

U = set of all strict preference relations on X.

A typical element of U is a (utility) function $u : X \to \mathbb{R}$

$U_A \subseteq U$, $U_B \subseteq U$

Example: $X = \{\kappa, \partial, \mu\}$. Then

$U = \{\kappa\partial\mu, \kappa\mu\partial, \partial\kappa\mu, \partial\mu\kappa, \mu\kappa\partial, \mu\partial\kappa\}$

U_A, U_B can be any subsets of U.

$\boxed{\text{Mechanism}}$ $g : S_A \times S_B \to X$.

$\boxed{\text{Game (of incomplete information) induced by the mechanism}}$

$v_A(s_A, s_B, u_A, u_B) = u_A(g(s_A, s_B))$, $s_A \in S_A$, $s_B \in S_B$

$v_B(s_A, s_B, u_A, u_B) = u_B(g(s_A, s_B))$

An equilibrium of this game is a pair (σ_A, σ_B) of functions $\sigma_A : U_A \to S_A$, $\sigma_B : U_B \to S_B$ such that

.

$$v_A(\sigma_A(u_A), s_B, u_A, u_B) \geq v_A(s_A, s_B, u_A, u_B)$$
$$v_B(s_A, \sigma_B(u_B), u_A, u_B) \geq v_B(s_A, s_B, u_A, u_B)$$

$\forall s_A \in S_A$, $\forall s_B \in S_B$, $\forall u_A \in U_A$, $\forall u_B \in U_B$. Hence

$$\boxed{\begin{array}{ll} u_A\, g(\sigma_A(u_A), s_B) \geq u_A\, g(s_A, s_B) \quad \forall \text{ALL} & (1) \\ u_B\, g(s_A, \sigma_B(u_B)) \geq u_B\, g(s_A, s_B) & (2) \end{array}}$$

SOCIAL CHOICE FUNCTION = $f: U_A \times U_B \to X$

f is IMPLEMENTABLE BY A MECHANISM $\langle S_A, S_B, g \rangle$ if the game induced by the mechanism has an equilibrium $\langle \sigma_A, \sigma_B \rangle$ such that $f(u_A, u_B) = g(\sigma_A(u_A), \sigma_B(u_B))$

f is IMPLEMENTABLE if it is implementable by some mechanism

f is INCENTIVE COMPATIBLE (IC) if the game induced by the direct mechanism $\langle U_A, U_B, f \rangle$ has $\langle id, id \rangle$ as an equilibrium, i.e.

$\langle U_A, U_B \rangle$ is an equilibrium, i.e.

$$\boxed{\begin{aligned} u_A f(u_A, u_B) &\geq u_A f(\hat{u}_A, u_B) & & (IC_A) \\ & & \forall \text{ ALL} & \\ u_B f(u_A, u_B) &\geq u_B f(u_A, \hat{u}_B) & & (IC_B) \end{aligned}}$$

REVELATION PRINCIPLE: f is IMPLEMENTABLE iff f is IC

<u>Proof</u>

Sufficiency: If f is IC, then f is implementable by the direct mechanism $\langle U_A, U_B, f \rangle$

Necessity: If f is implementable by $\langle S_A, S_B, g \rangle$ then (1) and (2) hold, and

$$f(u_A, u_B) = g(\sigma_A(u_A), \sigma_B(u_B)) \quad \forall u_A \, \forall u_B \qquad (3)$$

Equation (1), for $s_B = \sigma_B(u_B)$, $s_A = \sigma_A(\hat{u}_A)$, implies

$$u_A \, g(\sigma_A u_A, \sigma_B u_B) \geq u_A \, g(\sigma_A \hat{u}_A, \sigma_B u_B) \qquad (4)$$

(3) and (4) then imply (IC_A).

Similarly, (2) implies (IC_B).