Rationalizable (consumer) data

$$\text{DATA} = F = \{(p^t, x^t) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_+ : t = 1, \ldots, T\}.$$

AT PRICE VECTOR p^t, THE CONSUMER CHOSE x^t

T OBSERVATIONS IN ALL, T ANY NUMBER

IS THIS CONSUMER RATIONAL OR NOT?

RATIONAL = CONSISTENT WITH CONSUMER THEORY
||
RATIONALIZABLE

TESTABILITY OF CONSUMER THEORY

WE NEED TO IMPOSE AN ASSUMPTION ON PREFERENCES, OTHERWISE ALL DATA ARE CONSISTENT WITH CONSUMER THEORY.

EXAMPLE: $L = 2$, $T = 3$

Observation	p	x
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$
2	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$

WHEN IS A DATASET F CONSISTENT WITH

CONSUMER THEORY?

DEFINITIONS

FOR EACH OBSERVATION t, DEFINE

Budget set at observation t

$$B_t = \{ x \in \mathbb{R}^L_+ : p^t x \le \overbrace{p^t x^t}^{M} \} = B(p^t, x^t)$$

= BUDGET SET AT PRICES p^t AND INCOME $p^t x^t$

= SET OF ALL CONSUMPTION VECTORS THAT ARE AFFORDABLE AT PRICES p^t AND INCOME $p^t x^t$

= SOLUTION SET OF $L+1$ LINEAR INEQUALITIES

$$x_1 \geq 0,\ x_2 \geq 0,\ \quad ,\ x_L \geq 0$$

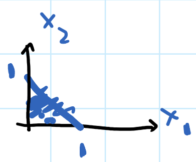
$$p^t x \le p^t x^t$$

= POLYHEDRON, HENCE CLOSED AND CONVEX

SINCE $p^t \gg 0$, $x^t \geq 0$, B_t IS ALSO BOUNDED

EXAMPLE

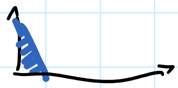
$$B_1 = \{ x \in \mathbb{R}^2_+ : x_1 + x_2 \le 1 \}$$

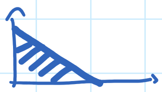


$$p^1 x \le p^1 x^1 \quad \Leftrightarrow$$

$$[1 \ \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le [1 \ \ 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$x_1 + x_2 \le 1, \quad x_1 \ge 0, \quad x_2 \ge 0$

$B_2 = \{x \in \mathbb{R}^2_+ : 3x_1 + x_2 \le 3\}$ 

$B_3 = \{x \in \mathbb{R}^2_+ ; 2x_1 + x_2 \le 3\}$ 

rationalizable

A DATASET F IS RATIONALIZABLE IF THERE EXISTS A LOCALLY NONSATIATED $\mathbb{R}^L_+ \xrightarrow{u} \mathbb{R}$ SUCH THAT FOR EACH $t = 1, 2, \ldots, T$

x^t MAXIMIZES u OVER B_t, OR EQUIVALENTLY

$\boxed{y \in B_t \Rightarrow u(x^t) \ge u(y)}$ ALL t

We will say that a utility function rationalizes the observed behavior (p^t, x^t) for $t = 1, \ldots, T$ if $u(x^t) \ge u(\mathbf{x})$ for all x such that $\mathbf{p}^t\mathbf{x}^t \ge p^t x$. That is, $u(\mathbf{x})$ rationalizes the observed behavior if it achieves its maximum value on the budget set at the chosen bundles. Suppose that the data were

To make the problem interesting, we have to rule out this trivial case. The easiest way to do this is to require the underlying utility function to be locally nonsatiated. Our question now becomes: what are the observable

RATIONALIZABLE = CONSISTENT WITH CONSUMER THEORY

Revealed preference relations

AFRIAT MATRIX

From the data set we can compute the square matrix A of order n defined by

$$a_{ij} := p_i \cdot (x_j - x_i) \text{ for all } i, j \in N.$$

A DATASET F DEFINES A TxT MATRIX $A = A(F)$, BY

$$A_{s,t} = p^s \cdot (x^t - x^s) \quad s, t \in [T]$$

$A_{s,t} \leq 0$ MEANS $p^s x^t \leq p^s x^s$

MEANS x^t IS AFFORDABLE FOR B_s

NOTE : $A_{tt} = 0 \quad \forall t$ (ZERO DIAGONAL)

USING THE AFRIAT MATRIX, WE ARE GOING TO DEFINE THREE RELATIONS ON THE SET OF OBSERVATIONS $[T] = \{1, 2, .., T\}$

Direct revealed preference

- ANY TxT MATRIX A DEFINES A RELATION ON [T]

$$\boxed{s \longrightarrow t \quad \text{iff} \quad A_{s,t} \leq 0 \quad \text{AND } s \neq t}$$

$s \rightarrow t$ READS :

s IS DIRECTLY REVEALED PREFERRED TO t.

$A_{s,t} \leq 0$ MEANS THAT x^t IS AFFORDABLE AT B_s

AND WE OBSERVED THE CONSUMER CHOOSE

x^s AT B_s

$s \to t$ THE CONSUMER CHOSE x^s EVEN IF HE COULD CHOOSE x^t

Example 1 : F is GIVEN BY

Observation	p	x
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$
2	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$

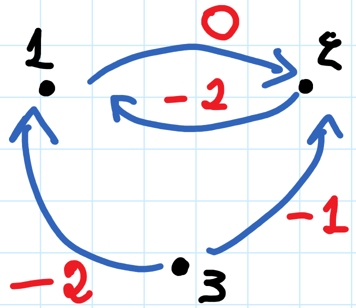
$$A_{23} = p^2 \cdot (x^3 - x^2) = [3 \quad 1] \cdot \left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} \right) =$$

$$= [3 \quad 1] \cdot \begin{bmatrix}0\\1\end{bmatrix} = 1$$

Afriat matrix

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} \end{array}$$

relation induced by F



$1 \xrightarrow{0} 2 \xrightarrow{-2} 1$, $2 \to 1 \to 2$

Revealed preference

LET $\xrightarrow{*}$ BE THE TRANSITIVE CLOSURE OF $\longrightarrow$

$s \xrightarrow{*} t$ IF THERE IS A PATH

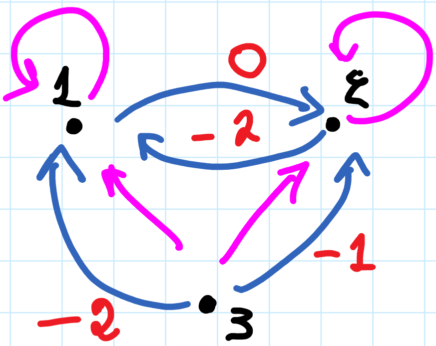
$s = s_0 \to s_1 \to s_2 \to \quad \to s_n = t$

FOR SOME $n \geq 1$, AND $s_i \in [T]$ $\forall i$

$s \xrightarrow{*} t$ READS:

S IS REVEALED PREFERRED TO t

IN THE EXAMPLE $1 \xrightarrow{*} 1$, WHILE NOT $1 \to 1$



TRANSITIVE CLOSURE OF $\longrightarrow$

$1 \xrightarrow{0} 2 \xrightarrow{-2} 1$

$2 \xrightarrow{+} 1$ because $2 \xrightarrow{-2} 1$

√√ $2 \xrightarrow{+} 2$ because $2 \xrightarrow{-2} 1 \xrightarrow{0} 2$

$3 \xrightarrow{+} 2$ because $3 \xrightarrow{-2} 1 \xrightarrow{0} 2$

F DOES NOT SATISFY GARP

Strict revealed preference

WE ALSO DEFINE THE RELATION $\xrightarrow{+}$ ON [T] by

$s \xrightarrow{+} t$ IF THERE $\mu, \nu \in [T]$ SUCH THAT

$s \xrightarrow{*} \mu \longrightarrow \nu \xrightarrow{*} t$ AND $A_{\mu\nu} < 0$

$$s \xrightarrow{*} p \longrightarrow v \xrightarrow{*} t, \text{ AND } A_{pv} < 0$$

$s \xrightarrow{+} t$ READS:

s IS REVEALED STRICTLY PREFERED TO t

SO $\xrightarrow{+}$ IS A SUBRELATION OF $\xrightarrow{*}$

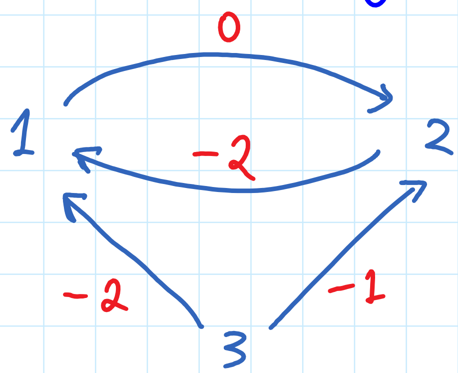
Example 1: DATASET F IS GIVEN BY

Observation	p	x
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$
2	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$

Afriat matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$$

relation induced by F



$3 \xrightarrow{+} 2$ because $3 \xrightarrow{-1} 2$

$3 \xrightarrow{-2} 1 \xrightarrow{0} 2$

$1 \xrightarrow{+} 2$ because $1 \xrightarrow{0} 2 \xrightarrow{-2} 1 \xrightarrow{0} 2$

$1 \to 2$ because $1 \to 2 \to 1 \to 2$

$2 \xrightarrow{+} 1$ because $2 \xrightarrow{-2} 1$

HENCE THE RELATION $\xrightarrow{+}$ IN THIS EXAMPLE HAS A CYCLE

$$1 \xrightarrow{+} 2 \xrightarrow{+} 1$$

EXAMPLE 2

Observation	p	x
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}4\\4\end{pmatrix}$
2	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$

Afriat matrix

$$A = \begin{bmatrix} 0 & -7 & -6 \\ 13 & 0 & 1 \\ 9 & -1 & 0 \end{bmatrix}$$

Revealed preference relation

$$\begin{tikzcd} 1 \arrow[rr, "-7"] \arrow[dr, "-6"'] & & 2 \\ & 3 \arrow[ur, "-1"'] & \end{tikzcd}$$

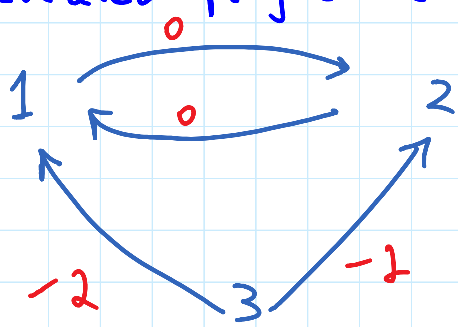
Example 3

Observation	p	x
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
2	$\begin{pmatrix} 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
3	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Afriat Matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ -2 & -1 & 0 \end{bmatrix}$$

Revealed preference relation



A CYCLE IN A RELATION IS A PATH THAT STARTS AND ENDS AT THE SAME POINT.

The data satisfy the ***Generalized Axiom of Revealed Preference****, abbreviated* ***GARP****, if no strict revealed preference* ***cycles*** *exist. That is, for no x^j is it the case that $x^j \succ^r x^j$.* KREPS

GENERALIZED AXIOM OF REVEALED PREFERENCE. *If $\mathbf{x}^t$ is revealed preferred to $\mathbf{x}^s$, then $\mathbf{x}^s$ cannot be strictly directly revealed preferred to $\mathbf{x}^t$.* VARIAN

GARP

THE DATASET F SATISFIES GARP IF THE INDUCED RELATION

Strict revealed preference $\xrightarrow{+}$ IS ACYCLIC

NAMELY, THERE IS NO PATH $s \xrightarrow{+} s$, FOR

ANY OBSERVATION S

HENCE THE DATASET OF EXAMPLE 2 DOES NOT SATISFY GARP

afriat theorem

Monday, January 17, 2022 09:56

Afriat's theorem. *Let* $(\mathbf{p}^t, \mathbf{x}^t)$ *for* $t = 1, \ldots, T$ *be a finite number of observations of price vectors and consumption bundles. Then the following conditions are equivalent.*

VARIAN P. 133

(1) There exists a locally nonsatiated utility function that rationalizes the data; F IS RATIONALIZABLE

(2) The data satisfy GARP; F SATISFIES GARP

(3) There exist positive numbers (u^t, λ^t) *for* $t = 1, \ldots, T$ *that satisfy the Afriat inequalities:* (LINEAR)

u^s, u^t, λ^t = VARIABLES

$$u^s \leq u^t + \lambda^t \underbrace{\mathbf{p}^t(\mathbf{x}^s - \mathbf{x}^t)}_{A_{t,s}} \quad \textit{for all } t, s;$$

SATISFIABLE

(4) There exists a locally nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.

IT SUFFICES TO SHOW $(1) \to (2) \to (3) \to (4)$

because $(4) \to (1)$ by definition

Rationalizability implies GARP

Monday, January 17, 2022 11:42

Afriat's theorem. *Let* $(\mathbf{p}^t, \mathbf{x}^t)$ *for* $t = 1, \ldots, T$ *be a finite number of observations of price vectors and consumption bundles. Then the following conditions are equivalent.*

(1) There exists a locally nonsatiated utility function that rationalizes the data;

(2) The data satisfy GARP;

(1) IMPLIES (2)

IF THE DATA F ARE RATIONALIZABLE, THEN THEY SATISFY GARP

Rationalizability and direct revealed preference

Monday, January 17, 2022 09:53

SUPPOSE THE DATA F ARE rationalizable BY A LOCALLY NONSATIATED UTILITY FUNCTION u. THEN FOR ANY TWO OBSERVATIONS s, t

$s \rightarrow t$ IMPLIES $u(x^s) \geq u(x^t)$

DIRECT REVEALED PREFERENCE (ALGEBRA WITH NO ECONOMIC MEANING)

IMPLIES

ACTUAL PREFERENCE (ECONOMIC MEANING)

PROOF

LET A BE THE AFRIAT MATRIX INDUCED BY F

By definition of Direct revealed preference $s \rightarrow t$ MEANS

$A_{st} \leq 0$ AND $s \neq t$ ie

$p^s(x^t - x^s) \leq 0$ AND $s \neq t$ ie

$p^s x^t \leq p^s x^s$ ie

by Budget set at observation t $x^t \in B_s$

x^t IS AFFORDABLE AT B_s

x^s IS OPTIMAL AT B_s

rationalizable THEN IMPLIES $u(x^s) \geq u(x^t)$

SUPPOSE THE DATA F ARE rationalizable. THEN
FOR ANY TWO OBSERVATIONS s, t
$s \rightarrow t \rightarrow s$ IMPLIES $A_{st} = 0 = A_{ts}$
IF THERE IS A 2-CYCLE $s \rightarrow t \rightarrow s$ (ALGEBRA, NO ECONOMIC MEANING) THEN
x^t IS AS EXPENSIVE AS x^s AT p^s ($A_{st}=0$)
x^s IS AS EXPENSIVE AS x^t AT p^t ($A_{ts}=0$)

PROOF

$s \rightarrow t \rightarrow s$

Rationalizability and direct revealed preference AND $s \rightarrow t$, $t \rightarrow s$ IMPLY

$u(x^s) \geq u(x^t)$ (1)
$u(x^t) \geq u(x^s)$ (2)
BY (1) (2)
$u(x^s) = u(x^t)$ (3)
$s \rightarrow t$ MEANS $A_{st} \leq 0$
IE $p^s x^t \leq p^s x^s$ IE
$x^t \in B_s = B(p^s, x^s)$ (4)
x^t IS AFFORDABLE AT B_s

$t \rightarrow s$

SIMILARLY $A_{ts} \leq 0$ IMPLIES
$x^s \in B_t$ (4')
x^s IS AFFORDABLE AT B_t

rationalizable IMPLIES
x^t MAXIMIZES u OVER B_t (5)
x^s MAXIMIZES u OVER B_s (5')

AT PRICES p^s
BY (4) AND (5') AND (3)
BOTH x^s, x^t ARE AFFORDABLE AT B_s
x^s IS OPTIMAL AT B_s ✓✓ ✓
x^s AND x^t HAVE THE SAME UTILITY

x^t IS ALSO OPTIMAL OVER B_s
THEN BY LOCAL NONSATIATION, x^t EXHAUSTS INCOME AT p_s IE
$p^s x^t = p^s x^s$ IE
$A_{st} = 0$ (6)

AT PRICES p^t
BY (4') AND (5) AND (3)
BOTH x^s, x^t ARE AFFORDABLE
x^t IS OPTIMAL
x^s, x^t HAVE THE SAME UTILITY

x^s IS OPTIMAL OVER B_t
THEN BY LOCAL NONSATIATION,
x^s EXHAUSTS INCOME AT p^t, NAMELY
$p^t x^s = p^t x^t$ IE
$A_{ts} = 0$ (6')

Rationalizability implies GARP

Monday, January 17, 2022 10:31

IF THE DATA F ARE RATIONALIZABLE, THEN THEY SATISFY GARP

CONSIDER A CYCLE OF THE Revealed preference RELATION,

$$s = t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow \dots t_N = s$$

WE WILL SHOW THAT $A_{t_i, t_{i+1}} = 0 \quad \forall i$

AND THEREFORE $\xrightarrow{+}$ IS ACYCLIC, BECAUSE

AT NO POINT IN THE CYCLE WE HAVE $A_{\mu\nu} < 0$

AND THEREFORE THAT F SATISFIES GARP

BY DEFINITION OF Direct revealed preference.

$t_i \longrightarrow t_{i+1}$ IF $t_i \neq t_{i+1}$ AND $\underline{A_{t_i, t_{i+1}} \leq 0}$

HENCE BY DEFINITION OF AFRIAT MATRIX

$$p^{t_i}\left(x^{t_{i+1}} - x^{t_i}\right) \leq 0, \quad 1 \leq i \leq N-1$$

$$x^{t_{i+1}} \in B_{t_i}, \quad 1 \leq i \leq N-1 \qquad (1)$$

$x^{t_{i+1}}$ IS AFFORDABLE AT PRICES p^{t_i}

AND BY Rationalizability and direct revealed preference

$$u(x^{t_i}) \geq u(x^{t_{i+1}}) \qquad 1 \leq i \leq N-1 \qquad (2)$$

x^{t_i} IS BETTER THAN $x^{t_{i+1}}$

BY (2)

$$\underline{u(x^s)} = u(x^{t_1}) \geq u(x^{t_2}) \geq \quad \geq u(x^{t_{N-1}}) \geq u(x^{t_N}) = \underline{u(x^s)}$$

I.E.

$$u(x^{t_1}) = u(x^{t_2}) = \quad = u(x^{t_N}) = u(x^s) \qquad (3)$$

THEN RATIONALIZABILITY IMPLIES

x^{t_i} = GLOBAL MAX OF u OVER B_{t_i} (4)

By (3) $u(x^{t_i}) = u(x^{t_{i+1}})$

By (1) $x^{t_{i+1}} \in B_{t_i}$

IE $x^{t_{i+1}}$ IS ALSO GLOBAL MAX OF u OVER B_{t_i}. HENCE BY LOCAL NONSATIATION

$$p^{t_i} x^{t_{i+1}} = p^{t_i} x^{t_i}$$

IE $x^{t_{i+1}}$ EXHAUSTS INCOME AT PRICES p^{t_i}

BUT THEN

$$A_{t_i, t_{i+1}} = p^{t_i} (x^{t_{i+1}} - x^{t_i}) = 0 \quad \forall i$$

examples

Monday, January 17, 2022 12:37

Example 1

Observation	p	x
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$
2	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$

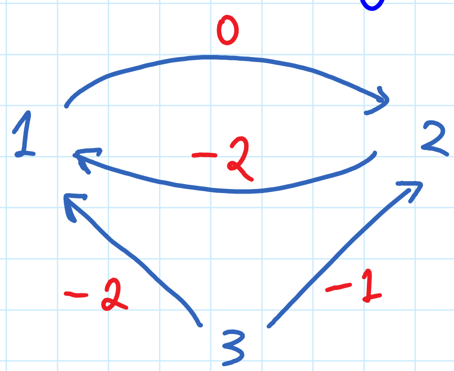
DATA

F

Afriat matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$$

relation induced by F



THERE IS A CYCLE

$$1 \xrightarrow{0} 2 \xrightarrow{-2} 1$$

WITH A NEGATIVE WEIGHT ON SOME EDGE

HENCE F DOES NOT SATISFY GARP, AND THEREFORE F IS NOT CONSISTENT WITH CONSUMER THEORY

EXAMPLE 2

Observation		x

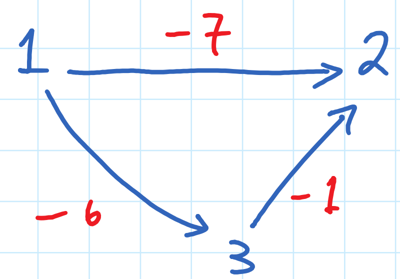
Observation	p	x
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}4\\4\end{pmatrix}$
2	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$

DATA
F

Afriat matrix

$$A = \begin{bmatrix} 0 & -7 & -6 \\ 13 & 0 & 1 \\ 9 & -1 & 0 \end{bmatrix}$$

Revealed preference relation



THERE ARE NO CYCLES, HENCE F SATISFIES GARP,
HENCE F IS CONSISTENT WITH CONSUMER THEORY

Example 3

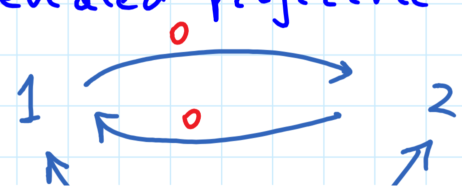
Observation	p	x
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$
2	$\begin{pmatrix}3\\3\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$

DATA
F

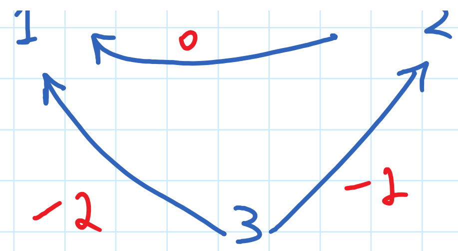
Afriat Matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ \cdots \end{bmatrix}$$

Revealed preference relation



$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ -2 & -1 & 0 \end{bmatrix}$$



THE ONLY CYCLE IS

$$1 \xrightarrow{0} 2 \xrightarrow{0} 1 \qquad 2 \xrightarrow{0} 1 \xrightarrow{0} 2$$

SINCE ALL WEIGHTS ON THE EDGES ARE ZERO $\xrightarrow{+}$ HAS NO CYCLES, F SATISFIES GARP AND THEREFORE F IS CONSISTENT WITH CONSUMER THEORY

A COMPUTER CAN TEST GARP ON ANY FINITE SET OF DATA, EFFICIENTLY.

END OF THE PROOF (1) → (2)

RATIONALIZIBILITY → GARP.

QUESTIONS?

GARP implies satisfiability of Afriat inequalities

Monday, January 17, 2022 12:53

Afriat's theorem. *Let $(\mathbf{p}^t, \mathbf{x}^t)$ for $t = 1, \ldots, T$ be a finite number of observations of price vectors and consumption bundles. Then the following conditions are equivalent.*

~~*(1) There exists a locally nonsatiated utility function that rationalizes the data;*~~

(2) IMPLIES (3)

(2) The data satisfy GARP;

(3) There exist positive numbers (u^t, λ^t) for $t = 1, \ldots, T$ that satisfy the Afriat inequalities:

$$u^s \leq u^t + \lambda^t \mathbf{p}^t(\mathbf{x}^s - \mathbf{x}^t) \quad \text{for all } t, s;$$

IF THE DATA F SATISFY GARP, THEN THE AFRIAT INEQUALITIES

$$\boxed{\begin{array}{ll} U_j \leq U_i + \lambda_i A_{ij} & \text{ALL} \\ \lambda_i > 0, \quad U_i > 0 & i, j \in [T] \end{array}}$$

ARE SATISFIABLE (= SOLVABLE)

VARIABLES $U_1 \; U_2 \quad U_T$ $\qquad 2T$

$\lambda_1, \lambda_2 \quad \lambda_T$

INEQUALITIES $T^2 + T$

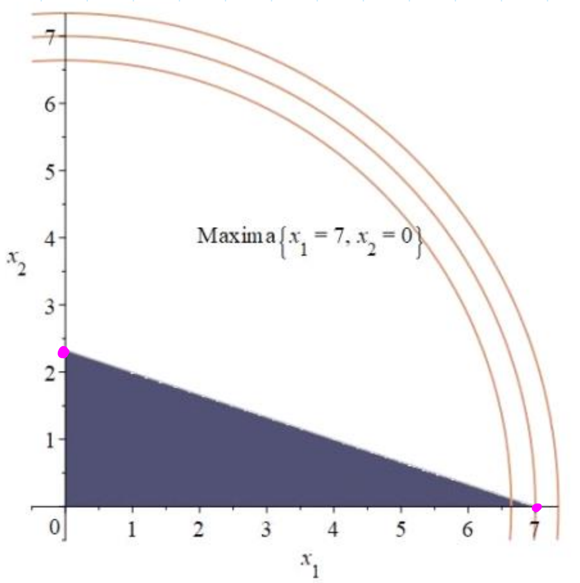
example

- TRUE (BUT UNOBSERVABLE) UTILITY FUNCTION $x_1^2 + x_2^2$
- TRUE (BUT UNOBSERVABLE) ENDOWMENT VECTOR $\omega = [1 \quad 2]$

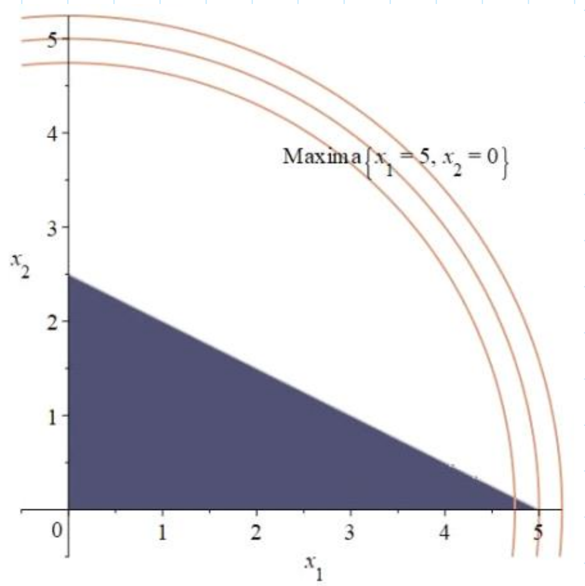
1. THE DATA ARE RATIONALIZABLE

- PRICE VECTORS [[1, 3], [1, 2], [2, 1]] $= [\, p^1 \quad p^2 \quad p^3 \,]$
- QUANTITY VECTORS [[7, 0], [5, 0], [0, 4]] $= [\, x^1 \quad x^2 \quad x^3 \,]$

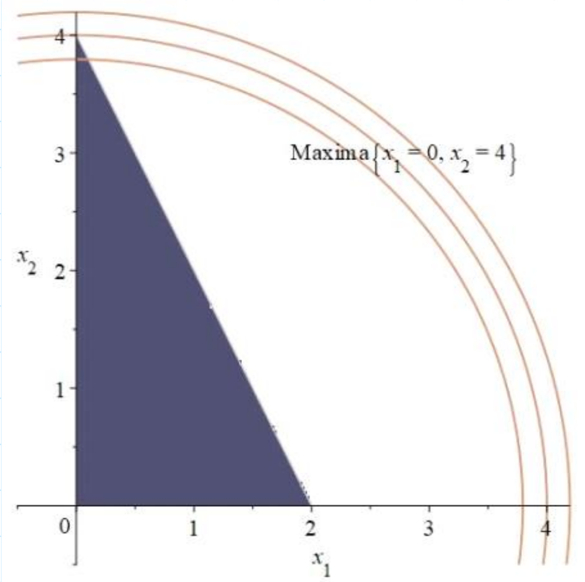
THE CORRESPONDING MAXIMIZATION DIAGRAMS



$p^1 = [1 \quad 3]$



$p^2 = [1 \quad 2]$



$p^3 = [2 \quad 1]$

DATASET $F = \{(p^1, x^1), (p^2, x^2), (p^3, x^3)\}$.

QUESTIONS ?

2. THE DATA SATISFY GARP

A = AFRIAT MATRIX OF F

$$A_{s,t} = p^s \cdot (x^t - x^s).$$

THE AFRIAT MATRIX DEFINES THE DIRECT REVEALED PREFERENCE RELATION ON THE SET OF OBSERVATIONS

$s \longrightarrow t$ IFF $s \neq t$ AND $A_{s,t} \leq 0$

WHICH MEANS: $p^s x^s \leq p^s x^t$

AT PRICES p^s, BOTH x^s AND x^t ARE AFFORDABLE, BUT THE CONSUMER CHOSE x^t INSTEAD OF x^s

$A_{st} < 0$ MEANS: $p^s (x^t - x^s) < 0$, $p^s x^t < p^s x^s$

AT PRICES p^s, x^t IS STRICTLY CHEAPER THAN x^s, BUT THE CONSUMER CHOSE THE MORE EXPENSIVE CHOICE x^s

CONSTRUCT THE AFRIAT MATRIX OF THE DATA

CONSTRUCT THE AFRIAT MATRIX OF THE DATA

$$A = \begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & 3 \\ 10 & 6 & 0 \end{bmatrix}$$

THE DIRECT REVEALED PREFERENCE RELATION CAN BE REPRESENTED BY A (WEIGHTED) GRAPH WHOSE NODES ARE THE OBSERVATIONS

THE DATA SATISFY GARP,
i.e.
THE STRICT REVEALED PREFERENCE RELATION is ACYCLIC,
i.e.

EITHER

1) THE REVEALED PREFERENCE RELATION is Acyclic

OR

2) EACH EDGE CONTAINED in A CYCLE of THE REVEALED PREFERENCE RELATION HAS WEIGHT 0

3. THE AFRIAT INEQUALITIES INDUCED BY THE DATA ARE SATISFIABLE

(3) There exist positive numbers (u^t, λ^t) for $t = 1, \ldots, T$ that satisfy the Afriat inequalities:

$$u^s \leq u^t + \lambda^t \overbrace{\mathbf{p}^t(\mathbf{x}^s - \mathbf{x}^t)}^{A_{ts}} \quad \textit{for all } t, s;$$

VARIAN

Lemma 4.5. *Real numbers v^j and $\alpha^j > 0$ for $i = 1, \ldots, J$ can be found such that, for all i and j,*

A_{ji}

Lemma 4.5. *Real numbers v^i and $\alpha^i > 0$ for $i = 1, \ldots, J$ can be found such that, for all i and j,*

$$v^i \leq v^j + \alpha^j \overbrace{[p^j \cdot x^i - p^j \cdot x^j]}^{A_{ji}}. \qquad \text{KREPS} \qquad (4.1)$$

Afriat's original argument begins by asserting the existence of numbers $\phi_1, \ldots, \phi_n$, and $\lambda_1, \ldots, \lambda_n > 0$ that satisfy the following unusual system of linear inequalities (from now Afriat inequalities) A_{ij} SCARF

$$\phi_j \leq \phi_i + \lambda_i a_{ij}, \text{ for all } i, j \in N .$$

2. *There exist strictly positive numbers U_i, λ_i for $i \in N$ satisfying the system of linear inequalities* A_{ji} CRAMA

$$U_i \leq U_j + \lambda_j p_j (q_i - q_j) \quad \forall i, j \in N. \qquad (3)$$

Afriat inequalities

$$\boxed{\begin{array}{lll} U_i \leq U_j + \lambda_j A_{ji} & & \text{ALL } i,j \in [T] \\ U_i > 0 & & \text{ALL } i \in [T] \\ \lambda_i > 0 & & \text{ALL } i \in [T] \end{array}}$$

VARIABLES IN RED

$$\begin{bmatrix} U_1 \le U_2 + 2\lambda_2 \\ U_1 \le U_3 + 10\lambda_3 \\ U_2 \le U_1 - 2\lambda_1 \\ U_2 \le U_3 + 6\lambda_3 \\ U_3 \le U_1 + 5\lambda_1 \\ U_3 \le U_2 + 3\lambda_2 \\ 0 < U_1 \\ 0 < U_2 \\ 0 < U_3 \\ 0 < \lambda_1 \\ 0 < \lambda_2 \\ 0 < \lambda_3 \end{bmatrix}$$

SYSTEM OF T^2+T INEQUALITIES

FIND ANY SOLUTION OF THE AFRIAT INEQUALITIES

$$\{U_1 = 3, U_2 = 1, U_3 = 3, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1\}$$

4. CONSTRUCT THE AFRIAT UTILITY FUNCTION INDUCED BY A SOLUTION OF THE AFRIAT INEQUALITIES

ANY SOLUTION OF THE AFRIAT INEQUALITIES PROVIDES A UTILITY FUNCTION THAT RATIONALIZES THE DATA F

THE SOLUTION $\{U_1 = 3, U_2 = 1, U_3 = 3, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1\}$ INDUCES THE AFFINE FUNCTIONS $w_t(x) = U_t + \underbrace{\lambda_t p^t}\,(x - x^t)$, $t \in [T]$

$$w_1(x) = -4 + x_1 + 3x_2$$

$p^t >> 0$

$$w_2(x) = -4 + x_1 + 2x_2$$

$w_1(x) = -4 + x_1 + 3x_2$

$w_2(x) = -4 + x_1 + 2x_2$

$w_3(x) = -1 + 2x_1 + x_2$

$p^t >> 0$

$\lambda_t > 0$

THE AFRIAT UTILITY FUNCTION INDUCED BY THE DATA IS

$$W(x) = \text{MIN}\{W_1(x), W_2(x), \ldots, W_T(x)\}$$

LEONTIEF UTILITY FUNCTION

$$w(x) = \min(-4 + x_1 + 2x_2, -4 + x_1 + 3x_2, -1 + 2x_1 + x_2)$$

THE HYPOGRAPH OF W IS

$$\text{HYPO}(W) = \{(x, t) : t \le W(x)\}$$

$$t \le W(x) \iff t \le \text{MIN}\{W_1(x), W_2(x), \ , W_T(x)\}$$

$$\iff t \le W_i(x) \text{ FOR ALL } i$$

$$\iff (x, t) \in \text{HYPO}(W_i) \text{ FOR ALL } i$$

$$\iff (x, t) \in \bigcap_{i=1}^{T} \text{HYPO}(W_i)$$

HYPO(W_i) = HALF SPACE = SOLUTION SET OF A SINGLE LINEAR INEQUALITY = CONVEX SET HENCE

HYPO(W) = INTERSECTION OF HALF-SPACES

= CONVEX SET

IE W IS A CONCAVE FUNCTION AS THE LOWER ENVELOPE OF AFFINE FUNCTIONS

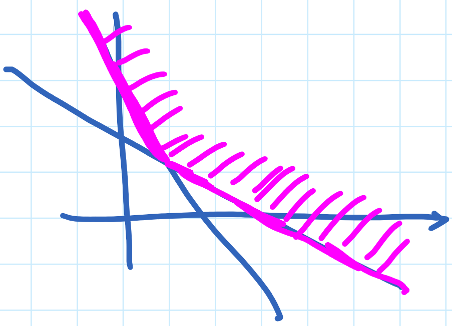
W(X) IS CONTINUOUS, STRICTLY INCREASING, CONCAVE.

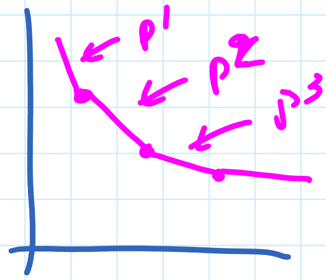
HOW DO THE INDIFFERENCE CURVES OF W LOOK LIKE?

$$B_c^W = \{ x \in R^L_+ : W(x) \geq c \}$$

$$= \{ x \in R^L_+ : \min_{i=1..T} \{ w_i(x) \} \geq c \}$$

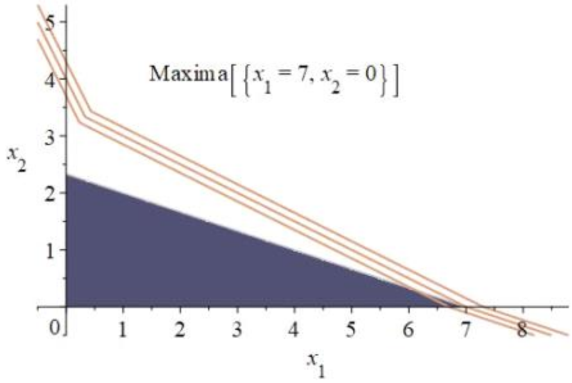
$$= \bigcap_{i=1}^{T} \{ x \in R^L_+ : w_i(x) \geq c \}$$





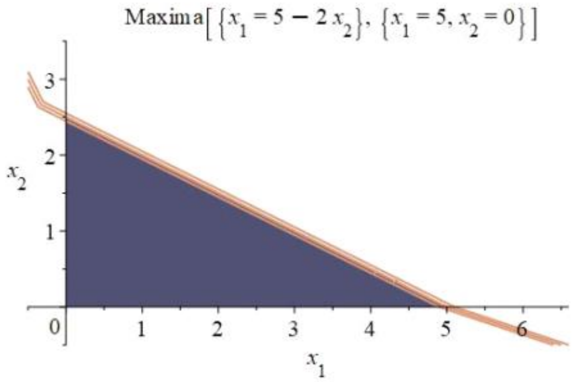
VERIFY THAT W RATIONALIZES THE DATA, IE SHOW THAT FOR EACH OBSERVATION t, THE VECTOR x^t IS A GLOBAL MAXIMUM OF

$$\begin{array}{l|l} \max\ W(x) & \\ p^t x \leq p^t \omega,\ x \geq 0 & t = 1, 2, \ldots, T \end{array}$$



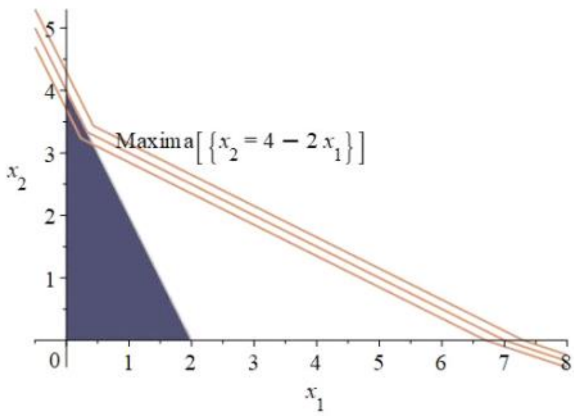
$p^1 = [1\ \ 3]$

$x^1 = [7\ \ 0]$



$p^2 = [1, 2]$

$x^2 = [5\ \ 0]$



$p^3 = [2\ \ 1]$

$x^3 = [0\ \ 4]$

END OF THE DESCRIPTION OF AFRIAT'S THEOREM. –

RECOVERABILITY OF PREFERENCES (IDENTIFICATION PROBLEM)

THE NONPARAMETRIC APPROACH TO DEMAND ANALYSIS

BY HAL R. VARIAN[1]

This paper shows how to test data for consistency with utility maximization, recover the underlying preferences, and forecast demand behavior without making any assumptions concerning the parametric form of the underlying utility or demand functions.

THE ECONOMIC THEORY of consumer demand is extremely simple. The basic behavioral hypothesis is that the consumer chooses a bundle of goods that is preferred to all other bundles that he can afford. Applied demand analysis typically addresses three sorts of issues concerning this behavioral hypothesis.

(i) Consistency. When is observed behavior consistent with the preference maximization model? W

(ii) Recoverability. How can we recover preferences given observations on consumer behavior? !

(iii) Extrapolation. Given consumer behavior for some price configurations how can we forecast behavior for other price configurations? !

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CONSISTENCY = GARP.

THE DATA ARE CONSISTENT WITH CONSUMER THEORY IF AND ONLY IF THEY SATISFY GARP. HENCE IF CONSUMER THEORY IS FALSE, TESTING IT FOR GARP ON SUFFICIENTLY LARGE DATASETS WILL EVENTUALLY REVEAL A VIOLATION OF GARP. IF, HOWEVER, CONSUMER THEORY IS TRUE, TWO PROBLEMS REMAIN

RECOVERABILITY (IDENTIFICATION PROBLEM):

NO AMOUNT OF DATA WILL RECOVER (IDENTIFY) THE TRUE UTILITY FUNCTION, NOT EVEN APROXIMATELY

(THE AFRIAT UTILITY FUNCTION IS FAR WAY FROM THE TRUE UTILITY FUNCTION)

AND THEREFORE ..

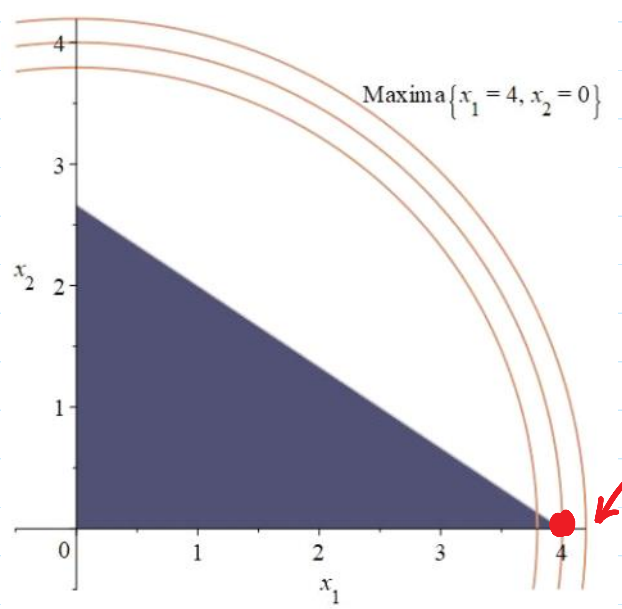
EXTRAPOLATIONS BASED ON ESTIMATED UTILITY FUNCTIONS MAY BE (BADLY) WRONG

EXTRAPOLATION EXAMPLE

WHAT WILL THE CONSUMER CHOOSE AT $p^4 = [4 \quad 6]$?

$\max u(x) = x_1^2 + x_2^2$

$p^4 x \leq p^4 w, \; x \geq 0$ $\qquad w = [1 \quad 2]$



$p^4 = [4 \quad 6]$

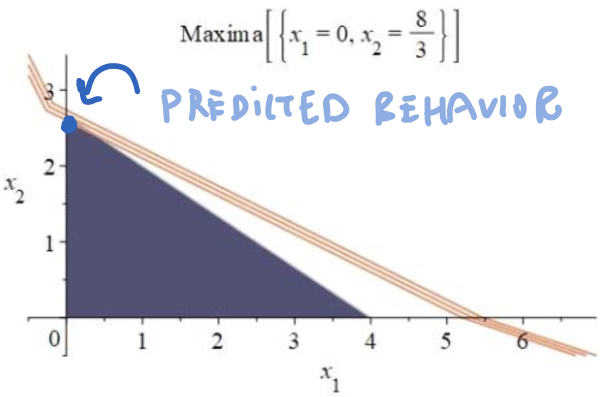
$x^4 = [4 \quad 0]$

TRUE BEHAVIOR

WHAT WILL BE THE EXTRAPOLATION (PREDICTION) OF THE ESTIMATED AFRIAT UTILITY FUNCTION ?

MAX $W(x)$

$p^4 x \le p^4 w, \; x \geq 0$ $\quad w = [1, 2]$



$p^4 = [4 \;\; 6]$

$x^4 = [0, \frac{8}{3}]$

NO AMOUNT OF EXTRA DATA WILL RECOVER THE TRUE UTILITY FUNCTION,

ALTHOUGH THE ESTIMATED AFRIAT UTILITY FUNCTIONS FITS THE DATA EXACTLY.

CONSUMER THEORY IS TOO WEAK, BUT TESTABLE (GARP)

PRODUCER THEORY IS TOO WEAK, BUT TESTABLE (WAPM)

$\{y_1 \quad y_t\} \subseteq Y \subseteq YO$

IF $f(x) = \min\{\psi_1(x), \ , \psi_k(x)\}$

THEN

$$\left.\begin{array}{l} \max f(x) \\ x \in S \end{array}\right|$$

IS EQUIVALENT TO

$$\left.\begin{array}{l} \max \ z \\ z \le \psi_1(x) \\ \vdots \\ z \le \psi_k(x) \\ x \in S \end{array}\right|$$

DIFFERENTIABLE

WE CAN APPLY

FRITZ JOHN

Afriat inequalities,simplified form

Afriat inequalities

$$\boxed{\begin{array}{ll} U_i \le U_j + \lambda_j A_{ji} & \text{ALL } i,j \in [T] \\ U_i > 0 & \text{ALL } i \in [T] \\ \lambda_i > 0 & \text{ALL } i \in [T] \end{array}}$$

SYSTEM OF $T^2 + 2T$ INEQUALITIES

$$U_i \le U_i + \lambda_i \cancel{A_{ii}}^{\,0} \qquad i = j$$

$$U_i \le U_i$$

Simplified afriat inequalities

$$\boxed{\begin{array}{ll} U_i \le U_j + \lambda_j A_{ji} & \text{ALL } i \ne j \in [T] \\ \lambda_i > 0 & \text{ALL } i \in [T] \end{array}}$$

SYSTEM OF T^2 INEQUALITIES

Satisfiable afriat lemma 1

ignore the requirement that each $U^i > 0$, as any solution to the remaining inequalities can be made to satisfy this simply by adding a large enough constant to each U^i. ~~We need to specify the matrix B that corresponds to~~

THE FOLLOWING ARE EQUIVALENT

$\boxed{1}$ THE Afriat inequalities ARE SATISFIABLE

$\boxed{2}$ THE Simplified afriat inequalities ARE SATISFIABLE

THE INEQUALITIES FOR $i = j$ READ

$$U_i \le U_i + \lambda_i A_{ii} \quad \text{i.e.} \quad U_i \le U_i$$

AND ARE THEREFORE REDUNDANT

$\boxed{1} \to \boxed{2}$ OK

$\boxed{E} \to \boxed{1}$

SUPPOSE $(U_i, \lambda_i)_{i=1}^{T}$ IS A SOLUTION OF THE

Simplified afriat inequalities . IF $U_i > 0$ FOR ALL i, DO NOTHING, ELSE

LET $\gamma = \min_i U_i \leq 0$. THEN $U_i \geq \gamma \;\; \forall i$

LET $\beta = 1 - \gamma$. THEN $\beta \geq 1$. DEFINE $U_i' = U_i + \beta$. THEN

$U_i' = (U_i - \gamma) + 1 \geq 1 > 0 \quad \forall i$

AND $(U_i', \lambda_i)_{i=1}^{T}$ IS A SOLUTION OF Afriat inequalities

BECAUSE

$U_i \leq U_j + \lambda_j A_{ji}$ IMPLIES

$U_i + \beta \leq U_j + \beta + \lambda_j A_{ji}$

Simplified weak afriat inequalities

$$\boxed{\begin{array}{ll} U_i \leq U_j + \lambda_j A_{ji} & \text{ALL } i \neq j \in [T] \\ \lambda_i \geq 1 & \text{ALL } i \in [T] \end{array}}$$

SYSTEM OF T^2 INEQUALITIES

Satisfiable afriat lemma,weak inequalities

THE FOLLOWING ARE EQUIVALENT

$\boxed{1}$ THE Simplified afriat inequalities ARE SATISFIABLE

$\boxed{E}$ THE Simplified weak afriat inequalities ARE SATISFIABLE

$\boxed{E} \to \boxed{1}$ OK

$\boxed{1} \to \boxed{2}$ LET $[U_i, \lambda_i]_{i=1}^{T}$ SOLVE THE Simplified afriat inequalities

LET $\gamma = \min \lambda_i$. THEN $\lambda_i \geq \gamma > 0$ FOR ALL i

DEFINE $\hat{\lambda}_i = \lambda_i / \gamma$, $\hat{U}_i = U_i / \gamma$

THEN $\hat{\lambda}_i \geq 1$, ALL i

AND $[\hat{U}_i, \hat{\lambda}_i]$ SOLVE THE Simplified weak afriat inequalities

BECAUSE

$$U_i \leq U_j + \lambda_j A_{ji} \quad \text{IMPLIES}$$

$$\frac{U_i}{\gamma} \leq \frac{U_j}{\gamma} + \frac{\lambda_j}{\gamma} A_{ji}$$

Garp implies Afriat matrix has a nonnegative row

SUPPOSE THE DATA F SATISFY GARP. THEN THE [AFRIAT MATRIX] A INDUCED BY F HAS AT LEAST ONE NONNEGATIVE ROW

Revealed strictly worse than relation

FOR EACH OBSERVATION i, $1 \le i \le T$, DEFINE

$C(i) = \{ j \in [T] : \text{THERE IS A PATH } i \xrightarrow{+} j \} =$

ALL j REVEALED PREFERRED STRICTLY WORSE THAN i

$n(i) = \# C(i)$

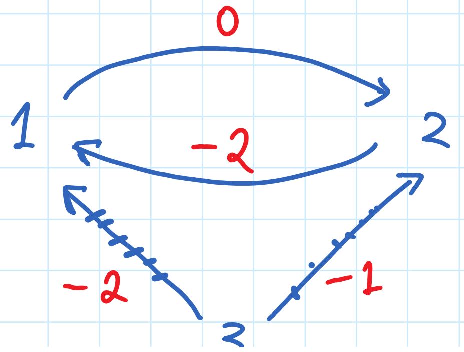
✓ $n(i) = 0$ MEANS THAT NO OBSERVATION IS REVEALED STRICTLY WORSE THAN i.

GARP MEANS $i \notin C(i) \quad \forall i$

$i \xrightarrow{+} i$ NO!

EXAMPLES

IN [Example 1] THE DIRECT RELATION INDUCED BY F IS

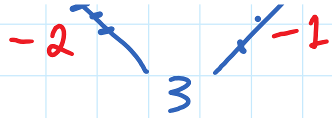


$C(1) = \{ j : 1 \xrightarrow{+} j \}$

$1 \xrightarrow{+} j$ there is a

PATH $1 \to \cdot \xrightarrow{<0} \cdot \quad \to \cdot j$

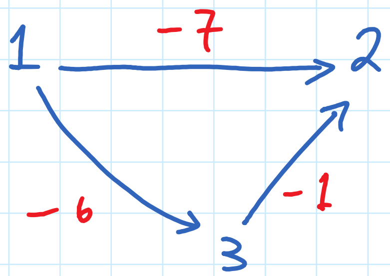
$1 \xrightarrow{-2} 2 \xrightarrow{0} 1$



$1 \xrightarrow{-2} 2 \xrightarrow{0} 1$

$C(1) = \{1\}$, $n(1) = 1$ | $C(2) = \{1, 2\}$, $n(2) = 2$ | $C(3) = \{1, 2\}$, $n(3) = 2$

IN EXAMPLE 2 THE DIRECT RELATION INDUCED BY F IS

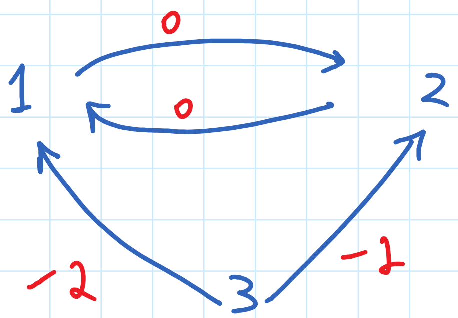


$C(1) = \{2, 3\}$, $n(1) = 2$

$C(2) = \emptyset$, $n(2) = 0$

$C(3) = \{2\}$, $n(3) = 1$

IN Example 3



$C(1) = \emptyset$ $n(1) = 0$

$C(2) = \emptyset$ $n(2) = 0$

$C(3) = \{1, 2\}$, $n(3) = 2$

Positive row lemma

SUPPOSE THE DATA F SATISFY GARP, AND A IS THE AFRIAT MATRIX INDUCED BY F. THEN

1. $n(i) < n(j)$ IMPLIES $A_{ij} > 0$
2. $n(i) = n(j)$ IMPLIES $A_{ij} \geq 0$
3. $n(i) = 0$ IMPLIES $ROW_i(A) \geq 0$

$\boxed{4}$ FOR AT LEAST ONE OBSERVATION i, $n(i) = 0$

NOTE $\boxed{3}$ AND $\boxed{4}$ IMPLY $Row_i(A) \geq 0$ FOR SOME i

PROOF

$\boxed{1}$ $\underbrace{n(i) < n(j)}$ IMPLIES $A_{ij} > 0$

MEANS THAT

THE NUMBER OF OBSERVATIONS REVEALED STRICTLY WORSE THAN j IS BIGGER THAN THE CORRESPONDING NUMBER FOR i.

$A_{ij} > 0$ MEANS $p^i(x^j - x^i) > 0$

IE $p^i x^j > p^i x^i$

IE x^j IS NOT AFFORDABLE AT B_i

SUPPOSE FOR CONTRADICTION THAT $\underbrace{n(i) < n(j)}$, $\underbrace{A_{ij} \leq 0}$

$0 \leq n(i) < n(j)$ AND THEREFORE

$n(j) \geq 1$, $\underbrace{C(j) \neq \emptyset}$

$i \neq j$

$i \neq j$ AND $A_{ij} \leq 0$ IMPLY $i \to j$

$C(j) \neq \emptyset$ IMPLIES THERE IS $\underbrace{k \in C(j)}$ IE

$i \xrightarrow{+} k$ AND THEREFORE

$j \xrightarrow{+} k$ AND THEREFORE

$i \to j \xrightarrow{+} k$ ie $i \xrightarrow{+} k$

ie $k \in C(i)$

WE HAVE SHOWN $k \in C(j)$ IMPLIES $k \in C(i)$

CONTRADICTION

$\boxed{2}$ $n(i) = n(j)$ IMPLIES $A_{ij} \geq 0$

SUPPOSE FOR CONTRADICTION $n(i) = n(j)$ AND $A_{ij} < 0$. THEN $i \xrightarrow{+} j$, hence $j \in C(i)$

AND $n(i) \geq 1$, THEN $n(j) = n(i) \geq 1$.

FOR ANY $k \in C(j)$ $j \xrightarrow{+} k$. AND THEREFORE

$i \xrightarrow{+} j \xrightarrow{+} k$ AND THEREFORE $k \in C(i)$. HENCE

$C(j) \subseteq C(i)$

SINCE F SATISFIES GARP, THERE IS NO

CYCLE $j \xrightarrow{+} j$ AND THEREFORE

$j \notin C(j)$

BY THE PURPLE FACTS $C(j) \subsetneq C(i)$

IE $n(j) < n(i)$, CONTRADICTION

$\boxed{3}$ $n(i) = 0$ IMPLIES $ROW_i(A) \geq 0$

↑

NO OBSERVATION
IS REVEALED
STRICTLY WORSE
THAN i

$n(i) = 0$ IMPLIES $n(j) \geq n(i)$ FOR ALL j

IF $n(i) < n(j)$ THEN $A_{ij} > 0$ BY $\boxed{1}$

IF $n(i) = n(j)$ THEN $A_{ij} \geq 0$ BY $\boxed{2}$

HENCE $ROW_i(A) \geq 0$

$\boxed{4}$ $n(i) = 0$ FOR AT LEAST ONE i

SUPPOSE, FOR CONTRADICTION, $n(i) \geq 1 \quad \forall i$

THIS IMPLIES THAT $C(i) \neq \emptyset$ FOR ALL OBSERVATIONS i

HENCE FOR EACH i, WE CAN CHOOSE $\mu(i)$ IN $C(i)$: $i \xrightarrow{+} \mu(i)$

$$1 \xrightarrow{+} \mu(1) \xrightarrow{+} \mu^2(1) \xrightarrow{+} \mu^3(1) \xrightarrow{+} \cdots$$

BUT THE NUMBER OF OBSERVATIONS IS FINITE (T)

HENCE THERE IS A REPETITION IN THE PATH, IE THERE EXIST NUMBERS $m < n$, $\mu^m(1) = \mu^n(1) = s$

$$s = \mu^m(1) \xrightarrow{+} \mu^{m+1}(1) \xrightarrow{+} \cdots \xrightarrow{+} \mu^n(1) = s$$

HENCE THE DATA DO NOT SATISFY GARP, CONTRADICTION

SUPPOSE FOR CONTRADICTION $n(i) \geq 1 \ \forall i$

THEN FOR EACH i THERE IS $\mu(i) \in C(i)$ IE

$i \xrightarrow{+} \mu(i) \longrightarrow \quad \longrightarrow$ ALL i $\longrightarrow$

$1 \leq i \leq T$, $1 \leq \mu(i) \leq T$

THEN

$$1 \xrightarrow{+} \mu(1) \xrightarrow{+} \mu^2(1) \xrightarrow{+} \mu^3(1) \xrightarrow{+} \cdots$$

SINCE $1 \leq \mu(i) \leq T \quad \forall i$

THERE IS A REPETITION ALONG THIS SEQUENCE $\xrightarrow{\text{IE}}$ $\exists\ m, n \quad m < n$

$\mu^m(1) = \mu^n(1) = s$ THEN

$$S = p^m(1) + p^n(1) = S$$

CONTRADICTING GARP

Base case, T=2

SUPPOSE THE DATA $F = \{(p^1, x^1), (p^2, x^2)\}$ SATISFY GARP.

THEN THE AFRIAT INEQUALITIES ARE SATISFIABLE, i.e

THERE EXIST $\hat{U}_1 > 0$, $\hat{U}_2 > 0$, $\hat{\lambda}_1 \geq 1$, $\hat{\lambda}_2 \geq 1$ THAT SATISFY

$$U_1 \leq U_2 + \lambda_2 A_{21}, \quad U_2 \leq U_1 + \lambda_1 A_{12}.$$

PROOF

GARP

$$A_{12} = 0 \Rightarrow A_{21} \geq 0 \qquad (1)$$
$$A_{12} < 0 \Rightarrow A_{21} > 0$$

A_{12} \ A_{21}	+	0	-
+	•	•	•
0	•	•	
-	•		

$$\underbrace{U_1 \leq U_2 + \lambda_2 A_{21}}, \quad U_2 \leq U_1 + \lambda_1 A_{12}.$$

ARE EQUIVALENT TO

(FOURIER- MOTZKIN ELIMINATION)

$$U_1 \leq U_2 + \lambda_2 A_{21}$$

$$U_2 - \lambda_1 A_{12} \leq U_1$$

$$U_\xi - \lambda_1 A_{12} \le U_1 \le U_\xi + \lambda_2 A_{21} \qquad (2)$$

ELIMINATING U_1

$$\underbrace{\cancel{U_\xi} - \lambda_1 A_{12} \le \cancel{U_2} + \lambda_2 A_{21}}$$

$$0 \le \lambda_1 A_{12} + \lambda_2 A_{21} \qquad (3)$$

CASE $A_{12} > 0$

SET $\hat\lambda_2 = 1$, $\hat\lambda_1 > 1$, $\underbrace{\hat\lambda_1 > -\frac{A_{21}}{A_{12}}}$, $\hat U_\xi > 0$

THEN $-\hat\lambda_1 A_{12} < A_{21}$ so

$$\hat U_\xi - \hat\lambda_1 A_{12} < \hat U_\xi + A_{21} = \hat U_2 + \hat\lambda_2 A_{21} \qquad \text{so choose}$$

$$\hat U_\xi - \hat\lambda_1 A_{1\xi} < \hat U_1 < \hat U_\xi + \hat\lambda_2 A_{21}$$

TO GET A SOLUTION OF THE AFRIAT INEQUALITIES

CASE $A_{12} = 0 = A_{21}$

THEN $\hat\lambda_1 = \hat\lambda_2 = 1$, $\hat U_1 = \hat U_\xi > 0$ WILL DO

CASE $A_{12} = 0$, $A_{21} > 0$

$$0 < \cancel{\lambda_1 A_{12}} + \lambda_2 A_{21} \quad \Bigg| \quad \hat\lambda_1 = 1 = \hat\lambda_2, \quad \hat U_\xi > 0$$

$$\boxed{\hat U_\xi - \cancel{\lambda_1 A_{12}} < U_1 < \hat U_2 + \hat\lambda_2 A_{21}}$$

$$\hat U_\xi < U_1 < \hat U_\xi + A_{21}$$

CASE $A_{12} < 0$

CASE $A_{12} < 0$

THEN $A_{21} > 0$

$\underbrace{0 < \lambda_1 A_{12} + \lambda_2 A_{21}}$ | $\hat{\lambda}_1 = 1$, $\hat{\lambda}_2 > -\frac{A_{12}}{A_{21}}$, $\hat{\lambda}_2 \geq 1$

$\hat{U}_\varepsilon > 0$

$$\boxed{\hat{U}_\varepsilon - \hat{\lambda}_1 A_{12} < U_1 < \hat{U}_2 + \hat{\lambda}_2 A_{21}}$$

$$\hat{U}_\varepsilon - A_{12} < \hat{U}_1 < \hat{U}_\varepsilon + \hat{\lambda}_2 A_{21}$$

IN ALL CASES, THERE EXISTS A SOLUTION S.T.

$A_{ji} \neq 0 \Rightarrow \hat{U}_i < \hat{U}_j + \hat{\lambda}_j \hat{A}_{ji}$

QUESTIONS ?

Induction step

SUPPOSE THAT GARP IMPLIES SATISFIABILITY OF THE AFRIAT INEQUALITIES FOR ALL DATASETS OF SIZE 1, 2, , T−1; SHOW THAT ANY DATA SET $F = \{(p^t\ x^t) : t = 1, \ T\}$ OF SIZE T, THE ASSOCIATED AFRIAT INEQUALITIES

$$\boxed{\begin{array}{ll} U_i \leq U_j + \lambda_j A_{ji} & \text{ALL } i \neq j \in [T] \\ \lambda_i \geq 1 & \text{ALL } i \in [T] \end{array}}$$

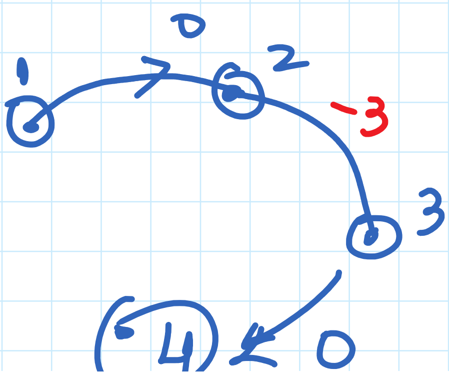
ARE SATISFIABLE.

PROOF

LET $F = \{(p^t, x^t) : t = 1 \quad T\}$ SATISFY GARP. DEFINE THE AFRIAT MATRIX A AND THE DIRECT REVEALED PREFERENCE RELATION $\longrightarrow$ OF F.

WE REPRESENT THE STRICT REVEALED PREFERENCE RELATION $\xrightarrow{+}$ USING SETS OF DESCENDANTS

ΣΥΜΒΟΛΑ ΑΠΟΓΟΝΩΝ



$2 \xrightarrow{+} 4$

4 IS A DESCENDANT OF 2

DESCENDANT OF

$(4) \leftarrow 0$

Σ

Garp implies Afriat matrices with a nonnegative row SHOWS THAT, IF WE DEFINE

$$C(i) = \{ j \in [T] : i \xrightarrow{+} j \}, \quad n(i) = \# C(i)$$

C(i) = STRICT DESCENDANTS OF i

n(i) = # C(i) = NUMBER OF DESCENDANTS OF i

$$\Gamma = \{ i \in [T] : n(i) = 0 \} \neq \emptyset \qquad (1)$$

Γ = NODES WITHOUT DESCENDANTS

$$i \in \Gamma \Rightarrow ROW_i(A) \geq 0 \qquad (2)$$

DEFINE $\Delta = \Gamma^c$

$$\Delta = \{ i \in [T] : n(i) \geq 1 \} \qquad (3)$$

= ALL OBSERVATIONS WITH AT LEAST ONE DESCENDANT.

CASE $\Delta \neq \emptyset$

Since $\Gamma \neq \emptyset$, WE HAVE $1 \leq \#(\Delta) \leq T-1$

THE DATA $\{ (p^t, x^t) : t \in \Delta \}$ SATISFY GARP

HENCE BY THE INDUCTION HYPOTHESIS

$$\boxed{\begin{array}{l} U_i \leq U_j + \lambda_j A_{ji} \\ \lambda_i \geq 1 \qquad i, j \in \Delta \end{array}} \quad \text{INEQ}(\Delta)$$

IS SATISFIABLE: LET $[\hat{U}_i, \hat{\lambda}_i : i \in \Delta]$ BE A SOLUTION OF INEQ(Δ). WE CAN ALWAYS ASSUME $\hat{U}_i > 0 \ \forall i \in \Delta$

WE ARE GOING TO EXTEND THE SOLUTION OF THE AFRIAT INEQUALITIES $[\hat{U}_i, \hat{\lambda}_i : i \in \Delta]$ TO A FULL SOLUTION $[\hat{U}_i, \hat{\lambda}_i : i \in [T]]$

Satisfiable afriat lemma 1

DEFINE θ BY

$$\theta = \text{MIN}\{\hat{U}_j + \hat{\lambda}_j A_{ji} : j \in \Delta,\ i \in \Gamma\} \quad (4)$$

WE EXTEND $[\hat{U}_i, \hat{\lambda}_i : i \in \Delta]$ TO A SOLUTION $[\hat{U}_i, \hat{\lambda}_i : i \in [T]]$ OF THE AFRIAT INEQUALITIES

$\checkmark$ $\hat{U}_i = \theta - \epsilon \qquad \underbrace{i \in \Gamma} \qquad (5) \checkmark$

$\checkmark$ $\hat{\lambda}_i = \underbrace{1} \qquad \underbrace{i \in \Gamma} \qquad (6) \checkmark$

$\epsilon > 0$

i \ j	Γ	Δ
Γ	??	?
Δ	?	OK BY INDUCTION HYPOTHESIS

CASE $i \in \Gamma$, $j \in \Gamma$

$\hat{U}_i \leq \hat{U}_j + \hat{\lambda}_j A_{ji}$

$\hat{\lambda}_i \geq 1$, $\hat{\lambda}_j \geq 1$

WANT TO SHOW

OR EQUIVALENTLY

$$\left.\begin{array}{l} \theta-\epsilon \le \theta-\epsilon + t A_{ji} \\ t \ge 1 \end{array}\right| \quad \begin{array}{l} 0 \le t A_{ji} \\ t \ge 1 \end{array}$$

WANT

$i, j \in \Gamma$ IMPLIES $n(i) = n(j) = 0$, THEREFORE BY

Positive row lemma $\qquad A_{ji} \ge 0 \qquad A_{ij} \ge 0$

HENCE THE INEQUALITIES ARE SATISFIED BY ANY $t \ge 1$

$i \backslash j$	Γ	Δ
	OK FOR ANY $t \ge 1$	??
	?	OK BY INDUCTION HYPOTHESIS

CASE $i \in \Gamma \quad j \in \Delta$

$$\left.\begin{array}{l} \hat{u}_i \le \hat{u}_j + \hat{\lambda}_j A_{ji} \\ \hat{\lambda}_i \ge 1 \quad \hat{\lambda}_j \ge 1 \end{array}\right|$$

WANT

THEN BY (5)

$$\left.\begin{array}{l} \theta-\epsilon \le \hat{u}_j + \hat{\lambda}_j A_{ji} \\ t \ge 1, \quad \hat{\lambda}_j \ge 1 \end{array}\right|$$

WANT

BY (4)

$$\theta = \min\{ \hat{U}_j + \hat{\lambda}_j A_{ji} : j \in \Delta,\ i \in \Gamma \}$$

$$\underbrace{\hat{U}_i = \theta - \epsilon}\ < \theta \le \underbrace{\hat{U}_j + \hat{\lambda}_j A_{ji}}$$

$i \backslash j$	Γ	Δ
Γ	OK FOR ANY $t \geq 1$	OK FOR ANY $t \geq 1$
Δ	?!	OK BY INDUCTION HYPOTHESIS

CASE $i \in \Delta$ $j \in \Gamma$

$$\left.\begin{matrix} \hat{U}_i \le \hat{U}_j + \hat{\lambda}_j A_{ji} \\ \hat{\lambda}_i \geq 1 \quad \hat{\lambda}_j \geq 1 \end{matrix}\right|\ \underline{\text{WANT}}$$

$\underbrace{i \in \Delta}$, $j \in \Gamma$ IMPLY

$$\underbrace{n(j) = 0} < \underbrace{1 \le n(i)} \qquad \underline{n(j) < n(i)}$$

Positive row lemma IMPLIES $A_{ji} > 0$

HENCE IT SUFFICES TO SHOW

$$\left.\begin{matrix} \hat{U}_i \le \theta - \epsilon + t \cdot \underbrace{A_{ji}} & i \in \Delta \\ \underbrace{\hat{\lambda}_i \geq 1,} \quad t \geq 1 & j \in \Gamma \end{matrix}\right|\ \text{WANT}$$

BY THE INDUCTION HYPOTHESIS IT SUFFICES TO SHOW

$$\left.\begin{matrix} \hat{U}_i - \theta + \epsilon \le t \cdot \underbrace{A_{ji}} \\ t \geq 1 \end{matrix}\right|\ \text{WANT}$$

$t \geq 1$ WANT

SINCE $A_{ji} > 0$, IT SUFFICES TO TAKE

$$t \geq \frac{\hat{u}_i - \theta + \epsilon}{A_{ji}} \qquad i \in \Delta$$

$$t \geq 1$$

$i \backslash j$	Γ	Δ
Γ	OK FOR ANY $t \geq 1$ ✓	OK FOR ANY $t \geq 1$ ✓
Δ	OK FOR t big ✓✓	OK BY INDUCTION HYPOTHESIS

CASE $\Delta = \emptyset$

THEN BY (1) $\Gamma = [T]$ i.e. $n(i) = 0 \ \forall i \in [T]$

Positive row lemma THEN IMPLIES $\mathrm{ROW}_i(A) \geq 0$ FOR ALL i,

HENCE $A \geq 0$. THEN $\hat{u}_i = \hat{\lambda}_i = 1$ ALL i

IS A SOLUTION OF THE AFRIAT INEQUALITIES

(3) There exist positive numbers (u^t, λ^t) for $t = 1, \ldots, T$ that satisfy the Afriat inequalities:

$$u^s \leq u^t + \lambda^t \mathbf{p}^t(\mathbf{x}^s - \mathbf{x}^t) \quad \text{for all } t, s;$$

(3) IMPLIES (4)

(4) There exists a locally nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.

LET $[\hat{U}_i, \hat{\lambda}_i]$ BE A SOLUTION OF THE AFRIAT INEQUALITIES, I.E.

$$\boxed{\begin{array}{ll} U_i \leq U_j + \lambda_j A_{ji} & \text{ALL } i,j \in [T] \\ U_i > 0 & \text{ALL } i \in [T] \\ \lambda_i > 0 & \text{ALL } i \in [T] \end{array}}$$

DEFINE $R^L_+ \xrightarrow{w_i} R$ BY

$$w_i(x) = \hat{U}_i + \hat{\lambda}_i p^i (x - x^i) \quad i \in [T]$$

THIS IS AN AFFINE FUNCTION. IT IS STRICTLY INCREASING BECAUSE $\hat{\lambda}_i > 0$ AND $p^i \gg 0$.

$$w(x) = \min\{w_1(x), \quad , w_T(x)\}$$

= LOWER ENVELOPE OF AFFINE FUNCTIONS

= CONCAVE + CONTINUOUS + STRICTLY INCREASING

SHOW THAT W RATIONALIZES THE DATA
IE FOR EACH OBSERVATION i, x^i
IS A GLOBAL MAX OF

$$\max \; w(x)$$

$$\underbrace{p^i x \leq p^i x^i}, \quad x \geq 0$$

SHOW $w(x^i) = \hat{U}_i \qquad i \in [T]$

$$w_i(x) = \hat{U}_i + \hat{\lambda}_i \, p^i \underbrace{(x - x^i)}$$

FOR $x = x^i$

$$w_i(x^i) = \hat{U}_i$$

$$w_j(x) = \hat{U}_j + \hat{\lambda}_j \cdot p^j (x - x^j)$$

$$w_j(x^i) = \hat{U}_j + \hat{\lambda}_j \underbrace{p^j (x^i - x^j)}$$

$$= \hat{U}_j + \hat{\lambda}_j A_{ji} \geq \hat{U}_i$$

$w_j(x^i) \geq \hat{U}_i$ FOR ALL $j \neq i$

$w_i(x^i) = \hat{U}_i$

$$w_i(x^i) = \hat{U}_i$$

$$w(x^i) = \text{MIN}\{w_1(x^i), w_2(x^i), \ , \ , w_T(x^i)\}$$

$$= \hat{U}_i$$

SHOW THAT x^i IS A GLOBAL MAXIMUM OF W OVER THE BUDGET SET $B_i = \{x \in R_+^L : \underline{p^i x \le p^i x^i}\}$

$\underline{x^i \in B_i}$, $\underbrace{w(x^i) = \hat{U}_i}$ (1)

FOR ANY OTHER $\underbrace{x \in B_i}$,

$$w(x) = \text{MIN}\{w_1(x) \qquad w_T(x)\}$$

$$w(x) \le w_i(x) = \underbrace{\hat{U}_i + \hat{\lambda}^i p^i(x - x^i)}$$

$$w(x) \le \hat{U}_i + \overset{\hat{\lambda}^i}{+} \underbrace{p^i(x - x^i)}_{\le 0} \le \hat{U}_i$$

$$w(x) \le \hat{U}_i \quad \text{FOR ALL } x \in B_i$$

$$w(x) \le w(x^i) \quad \text{FOR ALL } x \in B_i$$

IE x^i IS A GLOBAL MAX OF W OVER B_i

Revealed Preference and Aggregation

Laurens Cherchye,[*] Ian Crawford,[†] Bram De Rock[‡] and Frederic Vermeulen[§]

March 13, 2015

Abstract

In the tradition of Afriat (1967), Diewert (1973) and Varian (1982), we provide a revealed preference characterisation of the representative consumer. Our results are simple and complement those of Gorman (1953, 1961), Samuelson (1956) and others. They can also be applied to data very readily and without the need for auxiliary parametric or statistical assumptions.

4.D Aggregate Demand and the Existence of a Representative Consumer

MWG

The aggregation question we pose in this section is: When can we compute meaningful measures of aggregate welfare using the aggregate demand function and the welfare measurement techniques discussed in Section 3.I for individual consumers? More specifically, when can we treat the aggregate demand function as if it were generated by a fictional *representative consumer* whose preferences can be used as a measure of aggregate societal (or *social*) welfare?

Robert E. Lucas, Jr., (1987) proposed that the cost of business cycles be measured in terms of a proportional upward shift in the consumption process that would be required to make a representative consumer indifferent between its random consumption allocation and a nonrandom consumption allocation with the same mean. This measure of business cycles is the fraction Ω that satisfies

SARGENT

Whom or What Does the Representative Individual Represent?

My basic point in this paper is to explain that this reduction of the behavior of a group of heterogeneous agents *even if they are all themselves utility maximizers*, is not simply an analytical convenience as often explained, but is both unjustified and leads to conclusions which are usually misleading and often wrong. Why is this? First, such models are particularly ill-suited to studying macroeconomic problems like unemployment, which should be viewed as coordination

!

- EXISTENCE IN MODELS OF IMPERFECT COMPETITION.
- UNIQUENESS AND STABILITY FOR ALL KINDS OF MODELS

- WELFARE EVALUATIONS WHERE THE EVALUATION AGREES WITH THE UNDERLYING CONSUMER PREFERENCES.

Also, results in general equilibrium theory to be presented in ► Chap. 6 (the so-called Sonnenschein–Mantel–Debreu results) have shown that when the very restrictive assumptions guaranteeing the existence of a representative consumer are not satisfied, then market demand functions can differ drastically from the demand functions derivable from a single consumer and can have essentially any shape.

PCTRi

In conclusion, there seems to be very little reason to expect *market* demands to behave as if they came from a single consumer.[45] But unless they do, economic theory does not restrict the possible choice of functional forms for econometric estimation of market demand functions. Thus the relevance of Proposition

Econometrica, Vol. 45, No. 1 (January, 1977)

ON THE FOUNDATIONS OF THE THEORY OF MONOPOLISTIC COMPETITION

JU

BY JOHN ROBERTS AND HUGO SONNENSCHEIN[1]

Available theorems establishing the existence of general equilibrium in models incorporating imperfectly competitive firms rely on the assumption that reaction curves are continuous functions (or convex-valued, upper hemi-continuous correspondences). However, this property has not been derived from conditions on the fundamental data of tastes, technology, and maximizing behavior. We show here that continuity may fail even in extremely simple cases, with the result that equilibrium price and/or quantity choices fail to exist. The non-pathological nature of the examples we present suggests the need for a fundamental re-examination of the way our partial and general equilibrium models of monopolistic competition fit together.

WHAT IS A REPRESENTATIVE CONSUMER?

ΑΝΤΙΠΡΟΣΩΠΕΥΤΙΚΟΣ ΚΑΤΑΝΑΛΩΤΗΣ.

WE SAY THAT A (POSITIVE) REPRESENTATIVE CONSUMER EXISTS IN SOME GIVEN ECONOMY IF THE AGGREGATE DEMAND CORRESPONDENCE CAN BE OBTAINED

AS IF IT WERE DERIVED BY THE UTILITY MAXIMIZATION PROBLEM OF A SINGLE CONSUMER

A SAY THAT A NORMATIVE REPRESENTATIVE CONSUMER EXISTS IF

[1] A POSITIVE REPRESENTATIVE CONSUMER EXISTS, AND

[2] THE UTILITY FUNCTION OF THE POSITIVE REPRESENTATIVE CONSUMER MUST ALLOW FOR FAITHFUL WELFARE COMPARISONS

EVEN IF ALL CONSUMERS ACT ACCORDING TO CONSUMER THEORY, THE AGGREGATE DEMAND CORRESPONDENCE MAY NOT SATISFY GARP, I.E A REPRESENTATIVE CONSUMER MAY NOT EXIST

PROBLEM OF AGGREGATION, ACROSS CONSUMERS

Weak axiom of revealed preference(WARP)

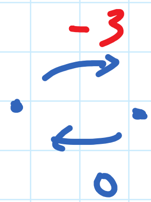
A DATA SET $F = \{(p^t, x^t) : t = 1, T\}$ SATISFIES WARP IF THE STRICT REVEALED PREFERENCE RELATION $\xrightarrow{+}$ INDUCED BY THE DATA F HAS NO CYCLES OF

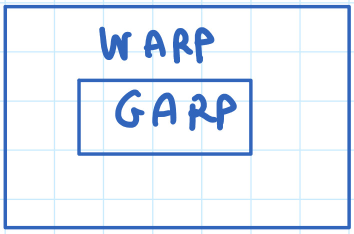
LENGTH 2, OR EQUIVALENTLY

$A_{ij} = 0$ IMPLIES $A_{ji} \geq 0$

$A_{ij} < 0$ IMPLIES $A_{ji} > 0$

FOR ALL $i \neq j$ IN $[T]$





Aggregate demand may fail WARP

CONSIDER AN EXCHANGE ECONOMY WITH THREE COMMODITIES AND TWO CONSUMERS, K and M, WE HAVE THE FOLLOWING DATA

$$p^1 = [3 \quad 2 \quad 1] \qquad p^2 = [2 \quad 3 \quad 2]$$

$$K^1 = [3 \quad 0 \quad 0] \qquad K^2 = [2 \quad 0 \quad 0]$$

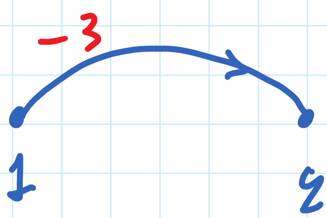
$$M^1 = [0 \quad 1 \quad 0] \qquad M^2 = [0 \quad 2 \quad 0]$$

STEP 1: THE BEHAVIOR OF CONSUMER K IS CONSISTENT THE THEORY OF THE CONSUMER

THE AFRIAT MATRIX OF THE DATA SET

$F_K = \{(p^1, K^1), (p^2, K^2)\}$ is

$$A_K = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$$



HENCE F_K SATISFIES GARP

A UTILITY FUNCTION THAT RATIONALIZES THE DATA IS

$$w^K(x) = \min\left(-5 + 3x_1 + 2x_2 + x_3, -5 + 3x_1 + \frac{9x_2}{2} + 3x_3\right)$$

STEP 2: THE BEHAVIOR OF CONSUMER M IS CONSISTENT WITH THE THEORY

THE AFRIAT MATRIX OF THE DATASET

$F_M = [(p^1, M^1), (p^2, M^2)]$ IS

$$A_M = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \qquad 1 \xleftarrow{-3} 2$$

HENCE F_M SATISFIES GARP

A UTILITY FUNCTION THAT RATIONALIZES THE DATA

$$w^M(x) = \min\left(-2 + 2x_1 + 3x_2 + 2x_3, -2 + \frac{9x_1}{2} + 3x_2 + \frac{3x_3}{2}\right)$$

STEP 3 THE DATASET

$F = [(p^1, K^1 + M^1), (p^2, K^2 + M^2)]$

DOES NOT SATISFY WARP, HENCE NO REPRESENTATIVE CONSUMER EXISTS FOR THIS ECONOMY

$K^1 + M^1 = [3 \quad 1 \quad 0] = x^1$

$K^2 + M^2 = [2 \quad 2 \quad 0] = x^2$

$A = \text{AFRIAT\_MATRIX}(F)$

$$A_{12} = p^1(x^2 - x^1) = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -1$$

$$A_{21} = p^2(x^1 - x^2) = \begin{bmatrix} 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -1$$

$$\begin{tikzcd} 1 \arrow[r, bend left, "-1"] & 2 \arrow[l, bend left, "-1"] \end{tikzcd}$$

VIOLATES WARP

SARP summary

Thursday, 13 December 2018 20:39

definitions

$B(p,M) = \{ y \in \mathbb{R}^L_+ : py \le M \}$ = budget set

WEAK AXIOM OF REVEALED PREFERENCE (WARP)

In any cycle $s \to t \to s$ of length one, $x^s = x^t$

ie $A_{st} \le 0$, $A_{ts} \le 0$ implies $x^s = x^t$, $A_{st} = 0 = A_{ts}$

STRONG AXIOM OF REVEALED PREFERENCE (SARP)

In any cycle $s = t_1 \to t_2 \to \dots \to t_{N-1} \to t_N = s$, σ

$x^{t_1} = x^{t_2} = \dots = x^{t_N} = x^s$

Preferences U are sufficiently convex if the problem

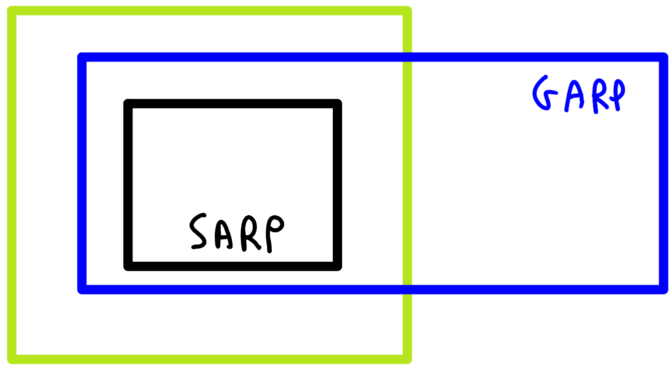
$\max \{ U(y) : py \le M, y \ge 0 \}$ has a unique solution

for any $p >> 0$ and any $M \ge 0$

- An observation (p, x) is STRONGLY RATIONALIZABLE if there exists a sufficiently monotonic and sufficiently convex utility function U such that x maximizes U over $B(p, px)$.

- A dataset F is STRONGLY RATIONALIZABLE if there exists a utility function U that strongly rationalizes each observation (p^t, x^t) in F

WARP


GARP
SARP

strong rationalizability implies SARP

Friday, 14 December 2018 16:29

If the dataset F is strongly rationalizable, then F satisfies SARP

Proof: F satisfies GARP by Rationalizability implies GARP, hence in any cycle

$s = t_1 \rightarrow \cdots \rightarrow t_N = s$ $\quad A_{t_i, t_{i+1}} = 0$, $U(x^{t_i}) = U(x^{t_{i+1}})$ $\forall i$

Hence $p^{t_i} x^{t_{i+1}} = p^{t_i} x^{t_i}$, ie $x^{t_{i+1}} \in B(p^{t_i}, p^{t_i} x^{t_i})$

ie $x^{t_{i+1}}$ is another maximizer of U over $B(p^{t_i}, p^{t_i} x^{t_i})$ ie

$x^{t_{i+1}} = x^{t_i}$ $\quad \forall i$ ie $x^s = x^{t_1} = \quad = x^{t_N} = x^s$, QED

$i \xrightarrow{++} j$ iff there is a path $i \xrightarrow{*} r \rightarrow v \xrightarrow{*} j$ $x^r \neq x^v$

Definition: For each observation i, $1 \leq i \leq T$ define

$C(i) = \{j : \exists$ path $i \xrightarrow{++} j\}$ = all j revealed strictly worse than i

$n(i)$ = number of elements in $C(i)$

Lemma 5

If A satisfies GARP then

(1) $n(i) < n(j)$ implies $A_{ij} > 0$

(2) $n(i) = n(j)$ implies $A_{ij} > 0$ or $x^i = x^j$

(3) $n(i) = 0$ implies $(A_{ij} > 0$ or $x^i = x^j)$ $\forall j$

(4) $n(i) = 0$ for at least one i

(3) and (4) imply that A has at least one nonnegative row, and that for each nonnegative row i, and for each j, $A_{ij} > 0$ or $x^i = x^j$

Proof

(1) Suppose for contradiction that $n(i) < n(j)$ and $A_{ij} \leq 0$ or $x^i = x^j$ (in which case $A_{ij} = A_{ji} = 0$).

Then for any $k \in C(j)$, $i \rightarrow j \xrightarrow{++} k$, ie $k \in C(i)$

ie $C(j) \subset C(i)$ ie $n(j) \leq n(i)$, contradiction

(2) Suppose for contradiction that $n(i) = n(j)$ and $A_{ij} < 0$

If $A_{ij} < 0$ then for any $k \in C(j)$ $i \xrightarrow{++} j \xrightarrow{++} k$, ie $j, k \in C(i)$
ie $C(j) \subset C(i)$ and $j \in C(i) \setminus C(j)$ ie $C(j) \subsetneq C(i)$
ie $n(j) < n(i)$, contradiction. Hence $A_{ij} \geq 0$.

Suppose for contradiction $A_{ij} = 0$ but $x^i \neq x^j$. Then
for any $k \in C(j)$ $i \xrightarrow{++} j \xrightarrow{++} k$ ie $j, k \in C(i)$ ie
$C(j) \subset C(i)$ and $j \in C(i) \setminus C(j)$ ie $C(j) \subsetneq C(i)$
ie $n(j) < n(i)$, contradiction

(3) $n(i) = 0 \Rightarrow n(i) \leq n(j) \; \forall j \underset{2}{\overset{!}{\Rightarrow}} (A_{ij} > 0 \text{ or } x^i = x^j) \; \forall j$

(4) Suppose for contradiction that $n(i) \geq 1 \; \forall i$. Then
there exists a path

$$1 \xrightarrow{++} \mu(1) \xrightarrow{++} \mu^2(1) \xrightarrow{++} \cdots$$

Since $1 \leq \mu^k(1) \leq T \; \forall k$, eventually there is a repetition
in this sequence, ie $\mu^i(1) = \mu^j(1)$ for some $j > i$. This
contradicts SARP

SARP implies strong afriat's inequalities

If A satisfies GARP then its STRONG AFRIAT INEQUALITIES

$$\boxed{\begin{array}{l} \alpha_j \le \alpha_i + \lambda_i A_{ij} \quad \forall i \forall j \\ x^i \ne x^j \Rightarrow \alpha_j < \alpha_i + \lambda_i A_{ij} \\ \lambda_i \ge 1 \qquad \forall i \\ \lambda_i P_i \ne \lambda_j P_j \quad \forall i \ne j \end{array}}$$

ARE SATISFIABLE

In particular, $x^i = x^j \Rightarrow \alpha_i = \alpha_j$

Proof by induction on T

Basis step: For $T=1$, $A = [0]$, so $\lambda_1 = 1$, $\alpha_1 = 0$ will do.

Induction step: Suppose the statement true for any dataset of size smaller than T, and let $F = \{(p^t, x^t) : t = 1 \dots T\}$ satisfy SARP. sarp implies Afriat matrices with a nonnegative row then implies that

$\Gamma = \{i : 1 \le i \le T, \eta(i) = 0\}$ is nonempty.

Define $\Delta = \{i : 1 \le i \le T, \eta(i) \ge 1\}$

Renumber the observations so that

$\Delta = \{1, \dots, K\}$, $\Gamma = \{K+1, \dots, T\}$, for some $0 \le K \le T-1$.

Case $\Delta = \emptyset$ Then $\eta(i) = 0 \; \forall i$ ie by lemma 5

$\forall i \forall j$ either $A_{ij} > 0$ or $x^i = x^j$

A solution of the Afriat inequalities is $\lambda_i \ge 1 \; \forall i$ and $\alpha_i = \theta \; \forall i$, for any $\theta \in \mathbb{R}$. Note that if $x^i \ne x^j$, then $A_{ij} > 0$, hence $\theta < \theta + A_{ij}$ ie $\alpha_j < \alpha_i + \lambda_i A_{ij}$

To guarantee $\lambda_i p^i \ne \lambda_j p_j \; \forall i \ne j$, let $E = \left\{ \frac{|p^i|}{|p^j|}, 1 \le i, j \le T \right\}$

E contains at most $2T^2$ distinct elements $e_{ij} = |p^i| / |p^j|$

Define $\lambda_1 = 1$, $\lambda_2 = 1 + \lambda_1 e_{12}$, $\lambda_3 = 1 + \lambda_1 e_{13} + \lambda_2 e_{23}$

and in general $\lambda_j = 1 + \sum_{v=1}^{j-1} \lambda_v e_{vj}$ for $j \geq 2$

Suppose, for contradiction, that $i<j$ and $\lambda_i p^i = \lambda_j p^j$. Then

$$\lambda_j = \lambda_i e_{ij} \quad \text{ie}$$

$$1 + \sum_{v=1}^{j-1} \lambda_v e_{vj} = \lambda_i e_{ij} \quad \text{ie}$$

$$1 + \sum_{v=1}^{i-1} \lambda_v e_{vj} + \cancel{\lambda_i} e_{ij} + \sum_{v=i+1}^{j-1} \lambda_v e_{vj} = \cancel{\lambda_i} e_{ij}$$

a contradiction

Case $\Delta \neq \emptyset$. Then $1 \leq \#(\Delta) \leq T-1$ since $\Gamma \neq \emptyset$.

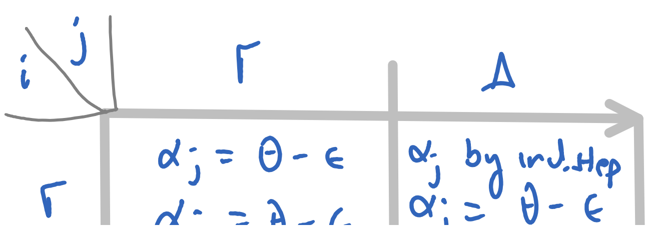
By the induction hypothesis, there exist $\alpha_i, \lambda_i, i \in \Delta$

$$\alpha_j \leq \alpha_i + \lambda_i A_{ij} \qquad \forall i \in \Delta \ \forall j \in \Delta$$

$$x^i \neq x^j \Rightarrow \alpha_j < \alpha_i + \lambda_i A_{ij} \qquad (1)$$

$$\lambda_i \geq 1$$

$$\lambda_i p_i \neq \lambda_j p_j \quad \forall i \neq j \text{ in } \Delta$$



SEE

Γ	$\alpha_j = \theta - \epsilon$ $\alpha_i = \theta - \epsilon$ $\lambda_i = t$	α_j by ind. step $\alpha_i = \theta - \epsilon$ $\lambda_i = t$
Δ	$\alpha_j = \theta - \epsilon$ α_i, λ_i by ind step	$\alpha_i, \alpha_j, \lambda_i$ by induction step

SEE BELOW

Let $$\theta = \min\left(\alpha_\delta + \lambda_\delta A_{\delta\gamma} : \delta \in \Delta, \gamma \in \Gamma\right) \qquad (2)$$

$$\alpha_j = \theta - \epsilon \quad \text{for some } \epsilon > 0, \quad j \in \Gamma \qquad (3)$$

$j \in \Gamma, \; i \in \Delta$

$$\alpha_j \overset{3}{=} \theta - \epsilon \overset{2}{<} \alpha_i + \lambda_i A_{ij}, \quad \text{QED}$$

$j \in \Delta, \; i \in \Gamma$

Then $n(i) = 0 < n(j)$ hence by lemma 4 $A_{ij} > 0$

Choose any $t \in R$ that satisfies $t > 1$

$$t > \max\left(\frac{\alpha_j - \alpha_i}{A_{ij}} : i \in \Gamma, j \in \Delta\right) \text{ and set}$$

$$\lambda_i = t + \sum_{v=1}^{i-1} \lambda_v e_{vi}, \quad i \in \Gamma = \{k+1, \dots, T\}$$

Then for any $i \in \Gamma$, $j \in \Delta$

$$\lambda_i \geq t > \frac{\alpha_j - \alpha_i}{A_{ij}}, \quad t > 1 \quad \text{so}$$

$\alpha_j < \alpha_i + \lambda_i A_{ij}$, $\lambda_i \geq 1$ QED

To show $\lambda_i p^i \neq \lambda_j p^j$ when $i \in \Gamma$, $j \in \Delta$, suppose for contradiction that $\lambda_i p^i = \lambda_j p^j$. Then

$\lambda_i = \lambda_j e_{ji}$ $i \in \Gamma$, $j \in \Delta$ ie

$$t + \sum_{v=1}^{i-1} \lambda_v e_{vi} = \lambda_j e_{ji} \quad \text{ie}$$

$$t + \sum_{v=1}^{j-1} \lambda_v e_{vi} + \cancel{\lambda_j e_{ji}} + \sum_{v=j+1}^{i-1} \lambda_v e_{vi} = \cancel{\lambda_j e_{ji}}$$

a contradiction

$j \in \Gamma$, $i \in \Gamma$

Then $n(i) = n(j) = 0$, so lemma 5 implies $A_{ik} \geq 0$, $A_{jk} \geq 0$ $\forall k$

$A_{ik} > 0$ or $x^i = x^k$ $\forall k$

$A_{jk} > 0$ or $x^j = x^k$ $\forall k$

hence in particular for the pair (i, j)

either $x^i = x^j$ or $(A_{ij} > 0$ and $A_{ji} > 0)$

If $x^i = x^j$ then $A_{ij} = A_{ji} = 0$ so $\alpha_j \leq \alpha_i + \lambda_i A_{ij}$ since $\alpha_i = \alpha_j = \theta - \epsilon$

If $x^i \neq x^j$ then $A_{ij} > 0$ and $A_{ji} > 0$, so

$\alpha_j < \alpha_i + \lambda_i A_{ij}$ since $\alpha_j = \alpha_i = \theta - \epsilon$, $\lambda_i = t > 1$

QED

To show $\lambda_i p^i \neq \lambda_j p^j$ $\forall$ $i<j$ in Γ, suppose for contradiction $\lambda_i p^i = \lambda_j p^j$. Then

$\lambda_j = \lambda_i e_{ij}$ ie

$$t + \sum_{v=1}^{j-1} \lambda_{vj} e_{vj} = \lambda_i e_{ij}$$, etc (as before)

Strong afriat inequalities imply strong rationalizibility

Saturday, 15 December 2018 15:54

If the strong Afriat inequalities of F are satisfiable, then there exists a strictly increasing, strictly concave utility function $\mathbb{R}^L_+ \xrightarrow{U} \mathbb{R}$ that strongly rationalizes F.

Proof Take any solution $(\alpha_i, \lambda_i : i=1,\dots,T)$ of the strong Afriat inequalities

$$\alpha_j \leq \alpha_i + \lambda_i A_{ij} \quad \forall i \forall j \qquad (1)$$

$$x^i \neq x^j \Rightarrow \alpha_j < \alpha_i + \lambda_i A_{ij} \qquad (2)$$

$$\lambda_i \geq 1 \quad \forall i \qquad (3)$$

$$\lambda_i p_i \neq \lambda_j p_j \quad \forall i \neq j \qquad (4)$$

$$x^i = x^j \Rightarrow \alpha_i = \alpha_j \qquad (5)$$

By (2), there is $\epsilon_0 > 0$ st

$$x^i \neq x^j \Rightarrow \alpha_j + \epsilon_0 < \alpha_i + \lambda_i A_{ij} \qquad (2')$$

For any $Q > 0$ define $\mathbb{R}^L \xrightarrow{g} \mathbb{R}$ by

$$g(x) = \left(|x|^2 + Q\right)^{1/2} - Q^{1/2} \qquad (6)$$

Then

$$x \neq 0 \Leftrightarrow g(x) > 0 \qquad (7)$$

$$x = 0 \Leftrightarrow g(x) = 0$$

$$\frac{\partial g}{\partial \dots} = \frac{1}{2}\left(|x|^2 + Q\right)^{-1/2} 2x_i = \frac{x_i}{\dots} < 1 \quad \text{ie}$$

$$\frac{\partial g}{\partial x_i} = \frac{1}{2}\left(|x|^2 + Q\right)^{-1/2} 2x_i = \frac{x_i}{\left(|x|^2+Q\right)^{1/2}} < 1 \quad \text{ie}$$

$$\partial g / \partial x_i = \frac{x_i}{g(x)+Q^{1/2}} < 1 \qquad (8)$$

g is strictly convex $\qquad (9)$

To show g strictly convex, it suffices to show

$$g(x) > g(a) + (x-a)\, g'(a) \quad \forall x \neq a$$

let $h(x) = \left(|x|^2 + Q\right)^{1/2}$ then it suffices to show

$$h(x) > h(a) + (x-a)\, h'(a) \quad \forall x \neq a$$

Since $\partial h / \partial x_i = \dfrac{x_i}{h(x)}$ it suffices to show

$$h(x) > h(a) + \frac{(x-a)\,a}{h(a)} \quad \forall x \neq a \qquad \text{ie}$$

$$h(x)\,h(a) > h^2(a) + (x-a)\,a \quad \text{ie}$$

$$\left(|x|^2 + Q\right)^{1/2}\left(|a|^2 + Q\right)^{1/2} > |a|^2 + Q + xa - |a|^2 \quad \text{ie}$$

$$\left(|x|^2 + Q\right)\left(|a|^2 + Q\right) > (xa + Q)^2 \quad \forall x \neq a \quad \text{ie}$$

$$|x|^2|a|^2 + Q|x|^2 + Q|a|^2 + \cancel{Q^2} > (xa)^2 + \cancel{Q^2} + 2Q\,xa \text{, ie}$$

$$\left((|x|\,|a|)^2 - |xa|^2\right) + Q\left(|x|^2 + |a|^2 - 2xa\right) > 0$$

$\uparrow \geq 0$ by Cauchy-Schwartz $\qquad \uparrow$ $|x-a|^2 > 0$ by $x \neq a$

By (7) $x^i - x^j \Rightarrow g(x^i - x^j) > 0$. Let $\mu > 0$ be the maximum of these numbers, and $0 < \epsilon < \epsilon_0/\mu$

Then $\epsilon g(x^i - x^j) < \epsilon\mu < \epsilon_0 \Rightarrow$ by (2')

$$x^i \neq x^j \Rightarrow \alpha_j + \epsilon g(x^i - x^j) < \alpha_i + \lambda_i A_{ij}$$

ie

$$x^i \neq x^j \Rightarrow \alpha_j < \alpha_i + \lambda_i A_{ij} - \epsilon g(x^i - x^j) \qquad (2'')$$

Define $R^L \xrightarrow{\phi_i} R$ by

$$\phi_i(x) = \alpha_i + \lambda_i p^i(x - x^i) - \epsilon g(x - x^i) \qquad (10)$$

By (10) (9), ϕ_i is strictly concave (11)

$$\phi_i(x^i) = \alpha_i \qquad (12)$$

Define $R^L \xrightarrow{U} R$ $\quad U(x) = \min\{\phi_i(x) : 1 \leq i \leq T\}$ (13)

By (11) (13) U is strictly concave (14)

Show each ϕ_i is strictly increasing when ϵ is small. and therefore U is strictly increasing for ϵ small (15)

$$\phi_i(x) = \alpha_i + \lambda_i \sum_{t=1}^{L} p^i(t)\, x_t - \lambda_i p^i x^i - \epsilon g(x - x^i)$$

$$\frac{\partial \phi_i(x)}{\partial x_t} = \lambda_i p^i(t) - \epsilon g_t(x - x^i)$$

$$\overset{8}{>} \lambda_i p^i(t) - \epsilon$$

Hence choose $0 < \epsilon < \min\{\lambda_i p^i(t) : 1 \le i \le T,\ 1 \le t \le L\}$

Show $U(x^j) \geq \alpha_j \quad \forall j \qquad (16)$

If not $\exists j \quad U(x^j) < \alpha_j$ ie

$\alpha_j > U(x^j) = \min\{\phi_i(x^j) : 1 \le i \le T\}$ ie

$\alpha_j > \phi_i(x^j)$ for some i, contradicting $(2'')$

U strongly rationalizes F ie $\forall i$

$p^i y \le p^i x,\ y \ne x \Rightarrow U(y) < U(x^i) \qquad (17)$

Let $p^j y \le p^j x^j,\ y \ne x^i$

$$U(y) \le \phi_i(y) = \alpha_i + \lambda_i p^i(y - x^i) - \epsilon g(y - x^i)$$

$$\le \alpha_i - \epsilon g(y - x^i) \overset{7}{<} \alpha_i \overset{16}{\le} U(x^i)$$

Representative consumer

Wednesday, 19 December 2018 11:32

Let $\mathbb{R}^L_+ \xrightarrow{U_i} \mathbb{R}$, $i = 1, 2, \ldots, N$ be utility functions, with demand correspondences $\mathbb{R}^L_{++} \times \mathbb{R}_+ \xrightarrow{D_i} \mathbb{R}^L$

<u>Definition</u> A utility function $\mathbb{R}^L_+ \xrightarrow{U} \mathbb{R}$ with demand correspondence $\mathbb{R}^L_{++} \times \mathbb{R}_+ \xrightarrow{D} \mathbb{R}^L$ represents $(U_1 \ldots U_N)$ if

$$D\left(p, \sum_{i=1}^{N} M_i\right) = \sum_{i=1}^{N} D_i(p, M_i) \quad \forall M_i \geq 0, \ \forall p \gg 0$$

Such a U, if it exists, is called a representative consumer. Unlike representative firms, representative consumers do not exist in general

Nonexistence of the representative consumer

Consider the following price-quantity pairs

	$p^1 = [3\ 2\ 1]$	$p^2 = [2, 3, 2]$
B	$B^1 = [3\ 0\ 0]$	$B^2 = [\varepsilon\ 0\ 0]$
C	$C^1 = [0\ 1\ 0]$	$C^2 = [0\ \varepsilon\ 0]$

Afriat matrix of $B = \begin{bmatrix} 0 & -3 \\ \varepsilon & 0 \end{bmatrix}$, no cycles, satisfies GARP, hence there is a utility function U_B with demand correspondance D_B such that

$$D_B(p^1, 9) = B^1, \quad D_B(p^2, 4) = B^2 \qquad (1)$$

Afriat matrix of $C = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$, no cycles, satisfies GARP hence there is a utility function U_C with demand correspondence D_C such that

$$D_C(p^1, 2) = C^1, \quad D_C(p^2, 6) = C^2 \qquad (2)$$

Suppose, for contradiction, that there is a representative consumer U with demand correspondence D. Then

$$D(p^1, 11) = B^1 + C^1 = [3 \ \ 1 \ \ 0] = \Sigma^1 \qquad (3)$$

$$D(p^2, 10) = B^2 + C^2 = [2 \ \ 2 \ \ 0] = \Sigma^2 \qquad (4)$$

Since $(p^1, \Sigma^1), (p^2, \Sigma^2)$ are obtained from the max problem of a single consumer, they must satisfy GARP.

The Afriat matrix generated by (p^1, Σ^1) (p^2, Σ^2) is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$; it fails GARP because there is a cycle of negative length

$$\begin{tikzcd} 1 \arrow[r, bend left, "-1"] & 2 \arrow[l, bend left, "-1"] \end{tikzcd}$$

Revealed Preference and Afriat's Theorem

This chapter concerns a consumer who, we hypothesize, is solving the CP for a number of different prices and incomes. We observe the consumer's choices and ask, What can we say about this consumer? In particular, are her choices consistent with the standard model of preference-driven, utility-maximizing choice? What patterns can we expect to see in the choices she makes, as we (say) vary one price only or her level of income? The emphasis here is on what can be discerned from a finite number of actual choices; Chapter 11 concerns the entire array of choices the consumer might make, for every possible level of income and every possible set of prices.

4.1. An Example and Basic Ideas

The main point of this chapter is illustrated by the following example. Imagine a consumer who lives in a three-commodity world and makes the following three choices.

- When prices are (10, 10, 10) and income is 300, the consumer chooses the consumption bundle (10, 10, 10).
- When prices are (10, 1, 2) and income is 130, she chooses the consumption bundle (9, 25, 7.5).
- When prices are (1, 1, 10) and income is 110, she chooses the consumption bundle (15, 5, 9).

Are these choices consistent with the standard model of the CP, in which the consumer has complete and transitive preferences and solves the CP for each set of prices and income?

This question is somewhat artificial. The story of the CP is that the consumer makes a single consumption choice, at one time, for all time. How then could we observe three different choices that she makes? The best we can do is to suppose that we have posed a set of hypothetical questions to the consumer of the form, If prices were p and your income was y, what would you purchase?[1]

Setting this artificiality to one side, a trivial affirmative answer to the question is possible. Imagine a consumer who is indifferent among, say, all bundles that give her less than 1000 units of each of the three goods. Since at these three sets of prices, the incomes she has are insufficient to purchase any bundle with 1000 units of each good, any choices—in particular, the choices she has made—are consistent with utility maximization, as long as they respect her budget constraint, which these do. This trivial answer may seem fanciful, but the point is not. To falsify the standard model, we must be able to use the data to conclude that some bundle is strictly preferred to some other(s). Otherwise, complete indifference is consistent with any pattern of choice that satisfies feasibility.

One way we might proceed is to ask whether the choices observed are consistent with preference maximization for strictly convex preferences. If a consumer with strictly convex preferences chooses the bundle x^* when prices are p and income is y, then the consumer strictly prefers x^* to any other bundle x such that $p \cdot x \leq y$, since we know that with strictly convex preferences and a convex choice set, the chosen bundle is strictly preferred to all feasible alternatives.

We take a slightly different path in this chapter, asking whether the observed choices are consistent with preference maximization for *locally insatiable* preferences. Local insatiability gives us cutting power according to the following lemma.

Lemma 4.1. *Suppose a consumer with complete, transitive, and locally insatiable preferences* $\succsim$ *chooses the consumption bundle* x^* *facing prices* p *with income* y*. Then we know that* $x^* \succsim x$ *for all bundles* x *such that* $p \cdot x = y$*. And we know that* $x^* \succ x$ *for all bundles* x *such that* $p \cdot x < y$*.*

Proof. The first part is obvious: If $p \cdot x = y$, x is feasible. Since x^* is chosen, it must be at least as good as x. The second part uses local insatiability: If $p \cdot x < y$, local insatiability ensures that there is some bundle x' near enough to x so that $p \cdot x' \leq y$, with $x' \succ x$. This means x' is feasible; hence $x^* \succsim x'$. But then $x^* \succsim x' \succ x$ gives the desired conclusion.

■

Now back to the example. From the data given above, we calculate the cost of each of the three selected bundles at each of the three sets of prices. This is done for you in Table 4.1.

		Prices		
		(10,10,10)	(10,1,2)	(1,1,10)
	(10,10,10)	300	130	120
Bundle	(9,25,7.5)	415	130	109
	(15,5,9)	290	173	110

Table 4.1. Cost of three bundles at three sets of prices.

In each case, the bundle selected exhausts the income of the consumer. This is required for these choices to be consistent with local insatiability: A locally insatiable consumer always spends all of her income; if a consumer ever chooses a bundle that costs strictly less than the income she has available, she cannot be maximizing locally insatiable preferences.

Beyond this, the important things to note are:

- When (10, 10, 10) was chosen (at prices (10, 10, 10) and income 300), the bundle (15, 5, 9) could have been purchased with some money left over. Apparently, this consumer strictly prefers (10, 10, 10) to (15, 5, 9).
- At the second set of prices (10, 1, 2), since (10, 10, 10) and (9, 25, 7.5) both cost 130 and (9, 25, 7.5) was selected, the latter must be at least as good as (10, 10, 10).
- At the third set of prices (1, 1, 10), the bundle (9, 25, 7.5) costs 109, while (15, 5, 9) costs 110. And we are told that with income 110, the consumer chose (15, 5, 9). Hence, (15, 5, 9) $\succ$ (9, 25, 7.5).

The data tell us that (10, 10, 10) $\succ$ (15, 5, 9), that (9, 25, 7.5) $\succeq$ (10, 10, 10), and that (15, 5, 9) $\succ$ (9, 25, 7.5). We can string these three deductions from the data together in the order (10, 10, 10) $\succ$ (15, 5, 9) $\succ$ (9.25, 7.5) $\succeq$ (10, 10, 10), which by transitivity (if the consumer has complete and transitive preferences) tells us that (10, 10, 10) $\succ$ (10, 10, 10). These data are therefore inconsistent with consumer behavior based on the standard preference-maximization model with locally insatiable preferences. On the other hand, suppose the third piece of data was instead:

- At prices (1, 2, 10) and income 115, the bundle selected is (15, 5, 9).

Then we would have come to no negative conclusions. At the first set of prices and income, the bundles (10, 10, 10) and (15, 5, 9) are affordable, and as the first bundle is selected and the (15, 5, 9) does not exhaust the budget constraint, (10, 10, 10) is revealed to be strictly preferred to (15, 5, 9). At the second set of prices and income level, (10, 10, 10) and (9, 25, 7.5) are precisely affordable and (9, 25, 7.5) is selected, so it is revealed to be weakly preferred to (10, 10, 10). This is just as before. But now, at the third set of prices and income level, of the three bundles only (15, 5, 9) is affordable. Knowing that it is selected tells us nothing about how it ranks compared to the other two; it could well come at the bottom of the heap. In fact, the other two choices tell us that (15, 5, 9) *must* come bottom among these three; the data are consistent with preferences among the three bundles that have (9, 25, 7.5) $\succ$ (10, 10, 10) $\succ$ (15, 5, 9), as well as preferences where (9, 25, 7.5) ~ (10, 10, 10) $\succ$ (15, 5, 9).

Of course, this argument doesn't tell us for sure that these three pieces of data are consistent with locally insatiable preference maximization; we need locally insatiable preferences for *all* of R^3_+ that support these three choices. But it is not hard to imagine that we can fill in preferences consistent with these data. The main result of this chapter, Afriat's Theorem, shows that we can construct preferences supporting these choices that are complete, transitive, and locally insatiable, and, in addition, strictly increasing, convex, and continuous.

4.2. GARP and Afriat's Theorem

To generalize the example, three definitions are needed. The setting throughout is one with k commodities, so that consumption bundles lie in R^k_+, prices are from R^k_{++}, and income levels come from R_+.

Definition 4.2.

a. *Take any finite set of (feasible) demand data:* $x^1 \geq 0$ *chosen at* (p^1, y^1), $x^2 \geq 0$ *chosen at* (p^2, y^2), ..., *and* $x^J \geq 0$ *chosen at* (p^J, y^J), *where, in addition,* $p^j \cdot x^j \leq y^j$ *for each j. If* $p^i \cdot x^j \leq y^i$, *the data* ***reveal directly that*** x^i ***is weakly preferred to*** x^j, *written* $x^i \succsim^d x^j$. *And the data* ***reveal directly that*** x^i ***is strictly preferred to*** x^j, *written* $x^i \succ^d x^j$, *if* $p^i \cdot x^j < y^i$. *(The superscript d is for* directly.*) Note that* $x^i \succ^d x^j$ *implies* $x^i \succsim^d x^j$.

b. *Suppose that for some* x^i *and* x^j, *there is a chain of direct revelations of weak preferences that start with* x^i *and end with* x^j. *That is, for some* x^{i1}, ..., x^{im}, $x^i \succsim^d x^{i1} \succsim^d x^{i2} \succsim^d \ldots \succsim^d x^{im-1} \succsim^d x^{im} \succsim^d x^j$. *Then the data* ***indirectly reveal that*** x^i ***is weakly preferred to*** x^j, *written* $x^i \succsim^r x^j$. *If some one or more of the steps in the chain is a direct relevation of strict preference, the data* ***indirectly reveal that*** x^i ***is strictly preferred to*** x^j, *written* $x^i \succ^r x^j$. *(The superscript r is for* revealed.*) In this definition, we allow for the case in which no intervening steps are required;* $x^i \succsim^d x^j$ *implies* $x^i \succsim^r x^j$, *and* $x^i \succ^d x^j$ *implies* $x^i \succ^r x^j$.

c. *The data satisfy the* ***Generalized Axiom of Revealed Preference****, abbreviated* ***GARP****, if no strict revealed preference* ***cycles*** *exist. That is, for no* x^i *is it the case that* $x^i \succ^r x^i$.

Part c sometimes confuses students, so let me be explicit on two grounds. First, suppose that for some x^i, p^i, and y^i, $p^i \cdot x^i < y^i$. Then according to part a of the definition, $x^i \succ^d x^i$; hence by part b, $x^i \succ^r x^i$, and hence GARP is violated. In words, GARP is violated if any bundle chosen at given prices and income costs less at those prices than the level of income. Second, suppose $x^i \succsim^r x^j$ and, simultaneously, $x^j \succ^r x^i$, for some pair x^i and x^j. That is, there is a chain of revealed weak preferences from x^i to x^j and a chain of revealed weak preferences, at least one of which is also strict, from x^j back to x^i. Then according to part b of the definition, $x^i \succ^r x^i$ and $x^j \succ^r x^j$, and this is also true for any element in either of the two chains of revealed preference. The two chains join together in a cycle, so there is a chain going from any link in the chain back to that link, with one of the links direct strict preference. Satisfaction of GARP is equivalently stated as: No such cycle can be found in the data.

Proposition 4.3 (Afriat's Theorem). *If a finite set of demand data violates GARP, these data are inconsistent with choice according to locally insatiable, complete, and transitive preferences. Conversely, if a finite set of demand data satisfies GARP, these data are consistent with choice according to complete, transitive, strictly increasing (hence, locally insatiable), continuous, and convex preferences.*

Before giving the proof, two comments are in order.

1. GARP concerns weak and strict revealed preferences among the finite collection of bundles that are chosen. We need not compare chosen bundles with those that never are chosen. *No violations of GARP among the set of chosen bundles* is necessary and sufficient for standard (locally insatiable) preferences for *all* of R^k_+.

2. If the data contain a violation of GARP, then no locally insatiable, complete, and transitive preferences can rationalize or explain the data. But if the data satisfy GARP, then not only can we produce locally insatiable, complete, and transitive preferences, but preferences which *in addition* are strictly increasing, continuous, and convex. In other words, given a finite collection of demand data, we cannot falsify the hypothesis that the consumer's preferences are strictly increasing or continuous or convex without throwing away the entire model of choice by locally insatiable, complete, and transitive preferences. The three extra properties add no testable restrictions.

   Please be careful in interpreting this. This does not say that it is impossible to falsify strictly increasing or convex preferences empirically. (I'm unwilling to make a claim one way or the other about continuity; whether continuity can be tested empirically depends on your definition of a valid empirical test.) Suppose, for instance, I ask a consumer to rank order the three distinct bundles x, x', and $0.5x + 0.5x'$, and she says the convex combination is definitely the worst of the three. Then we know she doesn't have convex preferences. Suppose I ask her to rank order three distinct bundles x, x', and x'' where x' and x'' are both $\geq x$ and neither $x' \geq x''$ nor $x'' \geq x'$, and she says x' is worst of the three. Then we can reject the hypothesis that she has strictly increasing preferences (and even nondecreasing preferences), without (yet) rejecting local insatiability. The point is, these are not questions about market demand data. What is asserted here is that, with a finite collection of *market demand* alone, I can't reject the three properties without simultaneously rejecting that her preferences are complete, transitive, and locally insatiable.

The proof of Afriat's Theorem

The first "half" of the proposition is easy. If the data are generated from locally insatiable, complete, and transitive preferences $\succsim$, then $x^i \succ^d x^j$ implies $x^i \succ x^j$, and $x^i \succsim^d x^j$ implies $x^i \succsim x^j$. The argument is the one given in Lemma 4.1. Therefore, by standard transitivity properties of strict and weak preferences, $x^i \succ^r x^i$ implies $x^i \succ x^i$, which violates the asymmetry of strict preference.

The proof of the second half of the proposition is long and very technical. The proof I am about to give is due to Varian (1982). I am unaware of any other use for these proof techniques in economics; to my knowledge, they give you no technique that can be usefully transferred to any other situation you will enounter. Therefore, I think you can almost surely skip this proof without risk of missing something later on. On the other hand, if you are an aficionado of very elegant proofs, this is one to see. Assume throughout that we have J demand choices—$x^j \geq 0$ chosen at prices p^j with income y^j, such that $p^j \cdot x^j \leq y^j$, for $j = 1, ..., J$—that collectively satisfy GARP.

As we remarked informally a page ago, for each j, $p^j \cdot x^j = y^j$; if $p^j \cdot x^j < y^j$, then $x^j \succ^d x^j$ according to the definition, which is a violation of GARP.

Lemma 4.4. *For each i, let n(i) be the number of indices j such that* $x^i \succ^r x^j$.

a. *If* $n(i) < n(j)$, *then* $p^i \cdot x^i < p^i \cdot x^j$.

b. *If* $n(i) = n(j)$, *then* $p^i \cdot x^i \leq p^i \cdot x^j$.

c. *At least one i satisfies* $n(i) = 0$.

Proof. For both a and b, we prove the contrapositives. (a) If $p^i \cdot x^i \geq p^i \cdot x^j = y^j$, then $x^i \succeq^d x^j$ by definition. But then if $x^j \succeq^r x^k$ for any k, it follows that $x^i \succeq^r x^k$, and hence the set of indices k such that $x^j \succeq^r x^k$ is a subset of the indices such that $x^i \succeq^r x^k$; $n(j) \leq n(i)$ follows immediately.

(b) And if $p^i \cdot x^i > p^i \cdot x^j$, then $x^i \succ^d x^j$. We know that every k such that $x^j \succeq^r x^k$ also satisfies $x^i \succeq^r x^k$, and there is at least one k, namely j itself, such that $x^i \succeq^r x^j$ but not $x^j \succeq^r x^i$. (If $x^j \succeq^r x^i$, GARP is violated.) Hence $n(i) > n(j)$. The contrapositive to this that $n(i) \leq n(j)$ implies $p^i \cdot x^i \leq p^i \cdot x^j$, and b then follows as a special case.

(c) If $n(i) \geq 1$ for every i, then for each i we can produce another index j such that $x^i \succ^r x^j$. Starting from any i, this gives us a chain $x^i = x^{i1} \succ^r x^{i2} \succ^r x^{i3} \succ^r$ …. Since there are only J possible values for the bundles, this chain must eventually cycle, which would violate GARP.

■

Lemma 4.5. *Real numbers v^i and $\alpha^i > 0$ for $i = 1, \ldots, J$ can be found such that, for all i and j,*

$$v^i \leq v^j + \alpha^j[p^j \cdot x^i - p^j \cdot x^j]. \tag{4.1}$$

Proof. We use induction on J. The result is trivially true for $J = 1$. Suppose it is true for all sets of data of size $J - 1$ or less. Take a set of data of size J (with no violations of GARP), and (renumbering if necessary) let 1 through I be the indices with $n(i) = 0$. By Lemma 4.4c we know that $I \geq 1$. Therefore, the set of indices $I + 1, \ldots, J$ gives us $J - 1$ or fewer pieces of data (with no violations of GARP). (The case where $I = J$ is handled by an easy special argument.) Hence we can produce v^i and α^i as needed for i from $I + 1$ to J, and inequality (4.1) holds for i and j both from $I + 1$ to J.

We extend to a full set of v^i and α^i as follows. Set

$$v^1 = v^2 = \ldots v^I = \min_{i=1,\ldots,I;\, j=I+1,\ldots,J} v^j + \alpha^j[p^j \cdot x^i - p^j \cdot x^j].$$

By this definition, (4.1) will hold for i from 1 to I and j from $I + 1$ to J.

To get (4.1) for i from $I + 1$ to J and j from 1 to I, we use α^j. Note that by Lemma 4.4a, for such i and j, since $n(i) > 0$ and $n(j) = 0$, we know that $p^j \cdot x^i > p^j \cdot x^j$. Therefore, we can select (for each $j = 1, \ldots, I$) α^j large enough so that these strictly positive terms give us the desired inequalities.

Finally, Lemma 4.4b tells us that for i and j both from 1 to I, $p^j \cdot x^i \geq p^j \cdot x^j$. Therefore, since $v^i = v^j$, no matter what (positive) values we chose for α^j, we have (4.1). This completes the induction step and the proof of Lemma 4.5.

■

The rest is easy. Define

$$u(x) = \min_{i=1,\ldots,J} v^i + \alpha^i[p^i \cdot x - p^i \cdot x^i].$$

Note that u is the minimum of a finite set of strictly increasing, affine functions; hence u is strictly increasing, concave, and continuous. (Math facts: The (pointwise) minimum of a finite set of strictly increasing functions is strictly increasing. The minimum of a finite set of concave functions is concave. The minimum of a finite set of continuous functions is continuous. If you did not know these facts, prove them.)

From (4.1), $u(x^i) = v^i$. This is a simple matter of comparing (4.1) with the definition of u.

We are done once we show that u rationalizes the data. To do this, take any observation (x^j, p^j, y^j). Because GARP is satisfied, $p^j \cdot x^j = y^j$. We know that x^j gives utility v^j. And it is evident from the definition of u that for any x such that $p^j \cdot x \leq y^j = p^j \cdot x^j$,

$$u(x) = \min_{i=1,\ldots,J} v^i + \alpha^i[p^i \cdot x - p^i \cdot x^i] \leq v^j + \alpha^j[p^j \cdot x - p^j \cdot x^j] \leq v^j.$$

That does it.

■

WARP: The Weak Axiom of Revealed Preference

In many economic textbooks, the so-called Weak Axiom of Revealed Preference, or WARP, is discussed. It may be helpful to make (brief) connections with what we have done here.

The *Weak Axiom of Revealed Preference* says that if x^* is chosen at (p, y), then x^* is strictly preferred to any other bundle x such that $p \cdot x \leq y$. This is almost a special case of GARP. It is a *special* case because it refers only to direct revelation of preference. GARP, on the other hand, looks at chains of revealed preference. But it is only *almost* a special case because it is a bit stronger than local insatiability allows; following Lemma 4.1, we can conclude only that when x^* is chosen at (p, y), then x^* is strictly preferred to any other bundle x such that $p \cdot x < y$, and is weakly preferred to x if $p \cdot x = y$.

The difference comes about because we are augmenting the standard model of preference maximization with local insatiability; WARP "works" if we augment the standard model with the maintained hypothesis that solutions to the CP are always unique, for example, if preferences are strictly convex.

4.3. Comparative Statics and the Own-Price Effect

Comparative statics is a term used by economists for questions (and answers to those questions) of the form, How does some economic quantity change as we change underlying parameters of the situation that generates it? Much of the empirical content of economics lies in the comparative statics predictions it generates. If within a model we can show that quantity x must rise if parameter z falls, and if the data show a falling z accompanied by a falling x, then we reject the original model.

In terms of consumer demand, the natural comparative statics questions are: How does demand for a particular good change with changes in income, holding prices fixed? How does demand for a good change with changes in the price of some other good, holding all other prices and income fixed? And—the so-called *own-price effect*—how does the demand for one good change with changes in the price of that good, holding other things fixed?

Everyday experience indicates that the theory on its own will not have much to say about income effects. There are goods the consumption of which declines as the consumer's wealth increases, at least over some ranges—public transportation is a commonly cited example. And there are goods the consumption of which rises with the consumer's wealth—taxicab rides, or skiing trips to the Alps. Goods whose consumption falls with wealth are called *inferior goods*, while those whose consumption rises with wealth are called *superior*. Moreover, when the percentage of income expended on a good rises as wealth rises, the good is called a *luxury* good; nonluxury goods are called *necessities*.

Of course, most goods do not fall neatly into a single one of these categories. Demand for public transportation by a given consumer rises as the consumer moves away from improverishment, and then falls as the consumer moves toward being rich. Indeed, since demand for all goods must be zero when $y = 0$, only a good that is never consumed in positive levels could qualify for *always* being inferior. Hence while a *superior good* is one the consumption of which never falls with rising income, an *inferior good* is one where the level of consumption *sometimes* falls with rising income.

As for the effect on the consumption of commodity i of a change in the price of commodity j, there is (again) little the bare theory of preference maximization can tell us. Demand for nails falls as the price of lumber rises, and the demand for corn rises with increases in the price of wheat. Roughly speaking, nails and lumber are complementary goods, while corn and wheat are substitutes. (This is rough for reasons that are discussed in later chapters, when precise definitions will be given.)

The best hope for a strong comparative statics prediction from the standard theory concerns own-price effects; everyday experience suggests that a consumer will demand less of a good as its own price rises. This is so strongly suggested by most people's experiences that goods for which this is true are called *normal*, while goods that are not normal—the demand for which sometimes rises as the price of the good rises—are called *Giffen* goods (named for Scottish economist Sir Robert Giffen, to whom the notion is attributed by Alfred Marshall).

The question is, if we look at demand by a preference-maximizing consumer, will demand for a good inevitably fall as the price of that good rises, holding everything else fixed? The answer, which you probably know from intermediate microeconomics, is no. One can draw pictures of indifference curves that support an increase in the consumption of a good as its own price rises.

With Afriat's Theorem, we can rigorize these pictorial demonstrations. Fix prices p, income y, and demand x at these prices and income. Choose some commodity (index i), and let p' be a price vector where all the prices except for good i are the same as in p, and $p'_i > p_i$. Let x' be demand at p' and y. Since (assuming local insatiability) $p \cdot x = y$ and p' is greater than p, as long as $x_i > 0$, $p' \cdot x > y$. *As long as* $p' \cdot x' = y$*, it doesn't matter what* x' *is—in particular, it doesn't matter whether* $x'_i \leq x_i$ *or* $x'_i > x_i$*—GARP will not be violated by these two data points*. Afriat's Theorem tells us that convex, strictly increasing, and continuous preferences can be found to support the existence of a Giffen good. Indeed, if we have any finite sequence of demand data for a fixed income level y and a succession of prices that involve (successive) rises in the price of good i only, as long as the demanded bundles satisfy the budget constraint with equality, GARP will not be violated.

A positive result

Consider the following alternative comparative statics exercise. Ask the consumer for her choice at prices p and income y. Suppose x is her choice. Now replace p with p', where p' is the same as p, except that the price of good i has been strictly increased, and *simultaneously* replace y by $y' = p' \cdot x$. Let x' be the chosen bundle at p' and y'. Suppose $x'_i > x_i$.

Since x is feasible at (p', y') by construction, we know that x' must be weakly preferred to x. But at the same time,

$$p' \cdot x' = \sum_{j \neq i} p'_j x'_j + p'_i x'_i = \sum_{j \neq i} p'_j x_j + p'_i x_i = p' \cdot x.$$

Rewrite the inner two terms as

$$\sum_{j \neq i} p_j x'_j + p'_i x'_i = \sum_{j \neq i} p_j x_j + p'_i x_i,$$

invoking the fact that $p'_j = p_j$ for $j \neq i$. Since $p'_i > p_i$ and $x'_i > x_i$, we know that $(p'_i - p_i)x'_i > (p'_i - p_i)x_i$; subtract the larger left-hand term from the left-hand side of the previous display, and the smaller right-hand term from the right-hand side of the display, and we see that $p \cdot x' < p \cdot x$. Therefore, for locally insatiable preferences, x is strictly preferred to x'. Oops. This demonstrates the following formal result.

Proposition 4.6. *Suppose x is chosen by the consumer facing prices p and income y, and x' is chosen at prices p' and income $p' \cdot x$, where p' is p except for an increase in the price of good i. If these choices are made according to the standard model with locally insatiable preferences, then $x'_i \le x_i$.*

In other words, if we ask this pair of questions of a consumer and find the consumption of good i rising, we have refuted (for this consumer) the standard model, augmented with local insatiability.

Giffen goods must be inferior

Before commenting on the result just derived, let me gather up one more "fact."

Proposition 4.7. *Suppose i is a Giffen good for some preference-maxiziming consumer with locally insatiable preferences. That is, for some income level y, price vectors p and p' such that p is identical to p' except that $p_i < p'_i$, and consumption bundles x and x' such that x is chosen at (p, y), x' is chosen at (p', y), $x'_i > x_i$. Then good i must be (sometimes) inferior for this consumer. More specifically, $y' = p \cdot x' < y$, and if x'' is a choice by the consumer facing (p, y'), then $x''_i > x_i$.*

Proof. Since $p' \cdot x' = y$ and $x'_i > x_i \ge 0$, we know that $y' = p \cdot x' < y$. Now suppose x'' is a bundle chosen at (p, y'). (To be completely rigorous about this, we ought to have insisted on augmenting the standard model of complete and transitive preferences with local insatiability *and continuity*, the latter to ensure that some bundle is chosen at every price and income combination.) Comparing x' and x'', we have that x' is chosen at (p', y), and x'' is chosen at $(p, p \cdot x')$, where p is p' except for a reduction in the price of good i. By an argument similar to that in the proof of Proposition 4.6, we conclude that $x''_i \ge x'_i$. But $x'_i > x_i$ by assumption; therefore $x''_i > x_i$.

■

Discussion

Why are Giffen goods possible? How could the consumption of good i rise with increases in its price? Roughly, the reason is that when the price of good i rises, two things happen. The relative price of good i, relative to the prices of other goods, is increased. Our expectations that the consumption of good i will fall (or, at least, not rise) stems from this; as the relative price of good i rises, the consumer ought to substitute other goods for it. But also the "level of real wealth" of the consumer falls; her income y is no longer sufficient to purchase the bundle x that she chose before the rise in p_i. A poorer consumer may choose more of good i because good i is inferior, and this implicit income effect may overcome the effect of the increased relative price of good i.

Indeed, the first alleged instance of a Giffen good concerned potatoes in Ireland during the great potato famine: The shortage of potatoes caused the price of potatoes, the staple crop of the working class, to rise precipitously. This so impoverished the working class that their diet came to consist almost entirely of … potatoes; they could no longer afford to supplement potatoes with other goods. The effect was so strong, it was claimed, that they purchased more potatoes. (Careful empirical evidence has been offered to refute that this did in fact happen.)

Proposition 4.7 supports this intuitive explanation, by showing that if a good is Giffen, it must be inferior. Or, to put it the other way around, if the good is superior—if there is no chance that reduced income leads to an increase in its consumption—then it cannot be Giffen; a rise in its price cannot lead to a rise in its level of consumption.

And Proposition 4.6 pretty much clinches the argument. Recall how the comparative statics exercise worked. We began with prices p, income level y, and a choice x by the consumer. The price of good i was increased, giving new prices p'. This makes the consumer worse off in real terms—she can no longer afford x (if $x_i > 0$)—so to compensate her, we increase her wealth to $y' = p' \cdot x$, just enough so that she could purchase x if she wanted to. Now the income effect of lower real wealth is controlled for, leaving only the relative price effect, and the consumer must choose a bundle x' with no more of good i than before.

Compensating the consumer in this fashion—giving her enough income so that at the new prices she can purchase the bundle at the original prices—is called Slutsky compensation. We pick up the story of compensated demand in Chapter 10, but for now we conclude with a final proposition, which is left for you to prove.

Proposition 4.8. *For a consumer with locally insatiable, complete, and transitive preferences, suppose that x is chosen at prices (p, y), and x' is chosen at prices p' and income $p' \cdot x$, for any other price vector p'. Then* $(p' - p) \cdot (x' - x) \leq 0$.

Coming attractions

We are far from finished with the classic theory of consumer demand, but we are going to take a break from it for a while. My personal prejudices are to undertake further foundations of models of choice—under uncertainty, dynamic, and social—before finishing the story. You (or your instructor) may feel differently about this, in which case you may wish to move to Chapters 10 and 11, concerning the dual consumer's problem, Roy's identity, the Slutsky equations, and integrability. But if you do this, a warning: The mathematical developments in Chapters 10 and 11 build on methods first employed in the theory of the profit-maximizing firm, in Chapter 9. So you should probably tackle Chapter 9 before Chapters 10 and 11.

Bibliographic Notes

Afriat's Theorem is given in Afriat (1967). The proof given here is taken directly from Varian (1982). The axioms of revealed preference discussed here are applied as well in the literature to demand *functions*, full specifications of consumer demand for all strictly positive prices and income levels; this part of the literature will be discussed in Chapter 11.

Problems

■ *4.1. In a three-good world, a consumer has the Marshallian demands given in Table 4.2. Are these choices consistent with the usual model of a locally insatiable, utility-maximizing consumer?

Prices			Income	Demand		
p_1	p_2	p_3	y	x_1	x_2	x_3
1	1	1	20	10	5	5
3	1	1	20	3	5	6
1	2	2	25	13	3	3
1	1	2	20	15	3	1

Table 4.2. Four values of Marshallian demand.

■ 4.2. There are a few details to clean up in the proof of Afriat's theorem. First, show that the minimum over a finite set of concave functions is concave, the minimum over a finite set of strictly increasing functions is strictly increasing, and the minimum over a finite set of continuous functions is continuous. Second, show how to proceed if, in the proof of Lemma 4.5, you find that $n(i) = 0$ for all i, and (hence) $I = J$.

■ 4.3. For a two-good world, create an indifference curve diagram that shows the (theoretical) possibility of a Giffen good.

■ *4.4. Prove Proposition 4.8.

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[1] A different way to try to make the story realistic is to suppose (1) that the consumer shops, say, each week, (2) has a fixed budget for each week, and (3) has preferences that are weakly separable from one week to the next and that are unchanging from week to week. Then our three pieces of data could be the results of three weeks of shopping. But suppositions 2 and 3 are rather incredible.

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THE NONPARAMETRIC APPROACH TO DEMAND ANALYSIS

BY HAL R. VARIAN[1]

This paper shows how to test data for consistency with utility maximization, recover the underlying preferences, and forecast demand behavior without making any assumptions concerning the parametric form of the underlying utility or demand functions.

THE ECONOMIC THEORY of consumer demand is extremely simple. The basic behavioral hypothesis is that the consumer chooses a bundle of goods that is preferred to all other bundles that he can afford. Applied demand analysis typically addresses three sorts of issues concerning this behavioral hypothesis.

(i) Consistency. When is observed behavior consistent with the preference maximization model?

(ii) Recoverability. How can we recover preferences given observations on consumer behavior?

(iii) Extrapolation. Given consumer behavior for some price configurations how can we forecast behavior for other price configurations?

The standard approach to these questions proceeds by postulating parametric forms for the demand functions and fitting them to observed data. The estimated demand functions can then be tested for consistency with the maximization hypothesis, used to make welfare judgements, or used to forecast demand for other price configurations. This procedure will be satisfactory only when the postulated parametric forms are good approximations to the "true" demand functions. Since this hypothesis is not directly testable, it must be taken on faith.

In this paper I describe an alternative approach to the above problems in consumer demand analysis. The proposed approach is *nonparametric* in that it requires no *ad hoc* specifications of functional forms for demand equations. Rather, the nonparametric approach deals with the raw demand data itself using techniques of finite mathematics. In particular I will show how one can directly and simply test a finite body of data for consistency with preference maximization, recover the underlying preferences in a variety of formats, and use them to extrapolate demand behavior to new price configurations. Thus each of the issues of concern to demand analysis mentioned above is amenable to the nonparametric approach.[2]

1. TESTING FOR CONSISTENCY WITH THE MAXIMIZATION HYPOTHESIS

Let $p^i = (p_1^i, \ldots, p_k^i)$ denote the ith observation of the prices of some k goods and let $x^i = (x_1^i, \ldots, x_k^i)$ be the associated quantities. Suppose that we have n

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[2]Another concern of applied demand analysis is the issue of testing for restrictions on the form of the utility function or budget constraint such as homotheticity, separability, etc. I address these questions in Varian [**29**, **30**].

observations on these prices and quantities, (p^i, x^i), $i = 1, \ldots, n$. How can we tell if these observations could have been generated by a neoclassical, utility maximizing consumer?

DEFINITION: A utility function $u(x)$ *rationalizes* a set of observations (p^i, x^i), $i = 1, \ldots, n$, if $u(x^i) \geqq u(x)$ for all x such that $p^i x^i \geqq p^i x$.

At the most general level there is a very simple answer to the above question: any finite number of observations can be rationalized by the trivial constant utility function $u(x) = 1$ for all x. The real question is when can the observations be rationalized by a sufficiently well behaved *nondegenerate* utility function? The best results in this direction are due to Sydney Afriat [**1, 2, 3, 4, 5**].

AFRIAT'S THEOREM: *The following conditions are equivalent:*

(1) *There exists a nonsatiated utility function that rationalizes the data.*

(2) *The data satisfies "cyclical consistency"; that is,*

$$p^r x^r \geqq p^r x^s, \qquad p^s x^s \geqq p^s x^t, \qquad \ldots, \qquad p^q x^q \geqq p^q x^r$$

implies

$$p^r x^r = p^r x^s, \qquad p^s x^s = p^s x^t, \qquad \ldots, \qquad p^q x^q = p^q x^r.$$

(3) *There exist numbers* U^i, $\lambda^i > 0$, $i = 1, \ldots, n$, *such that*

$$U^i \leqq U^j + \lambda^j p^j (x^i - x^j) \qquad \textit{for} \quad i, j = 1, \ldots, n.$$

(4) *There exists a nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.*

PROOF: See Appendix 1.

There are several remarkable features of Afriat's theorem. First, the equivalence of (1) and (4) shows that if some data can be rationalized by any nontrivial utility function at all it can in fact be rationalized by a very nice utility function. Or put another way, violations of continuity, concavity, or monotonicity cannot be detected with only a finite number of demand observations. Secondly, the numbers U^i and λ^i referred to in part (3) of Afriat's theorem can be used to actually construct a utility function that rationalizes the data. The numbers U^i and λ^i can be interpreted as measures of the utility level and marginal utility of income at the observed demands. This is described in more detail in Appendix 1.

Thirdly, parts (2) and (3) of Afriat's theorem give directly testable conditions that the data must satisfy if it is to be consistent with the maximization model. Condition (3) for example simply asks whether there exists a nonnegative solution to a set of linear inequalities. The existence of such a solution can be checked by solving a linear program with $2n$ variables and n^2 constraints. Diewert and Parkan [**10**] describe some of their computational experience with this technique using actual demand data. Unfortunately the fact that the number of constraints rises as the square of the number of observations makes this condition difficult to verify in practice for computational reasons.[3]

Condition (2) seems rather more promising from the computational perspective. As it turns out, there is an equivalent formulation of condition (2) which is quite easy to test. In addition this equivalent formulation is much more closely related to the traditional literature on the revealed preference approach to demand theory of Samuelson **[24]**, Houthakker **[12]**, Richter **[21]**, and others. In order to describe this formulation we must first consider the following definitions:

DEFINITIONS: Given an observation x^i and a bundle x:

(1) x^i is *directly revealed preferred* to x, written $x^i R^0 x$, if $p^i x^i \geqq p^i x$.

(2) x^i is *strictly directly revealed preferred* to x, written $x^i P^0 x$, if $p^i x^i > p^i x$.

(3) x^i is *revealed preferred* to x, written $x^i R x$, if $p^i x^i \geqq p^i x^j$, $p^j x^j \geqq p^j x^l, \ldots, p^m x^m \geqq p^m x$ for some sequence of observations $(x^i, x^j, \ldots, x^m)$. In this case we say that the relation R is the *transitive closure* of the relation R^0.

(4) x^i is *strictly revealed preferred* to x, written $x^i P x$, if there exist observations x^j and x^l such that $x^i R x^j$, $x^j P^0 x^l$, $x^l R x$.

Note that in the above definitions we do *not* require x^i, x^j, x^l, etc. to be distinct observations. We also adopt the convention that xRx for all bundles x.

DEFINITIONS: A set of data satisfies the:

(1) *Strong Axiom of Revealed Preference, version* 1 (SARP 1) if $x^i R x^j$ and $x^j R x^i$ implies $x^i = x^j$;

(2) *Strong Axiom of Revealed Preference, version* 2 (SARP 2) if $x^i R x^j$ and $x^i \neq x^j$ implies not $x^j R x^i$;

(3) *Strong Axiom of Revealed Preference, version* 3 (SARP 3) if $x^i R x^j$ and $x^i \neq x^j$ implies not $x^j R^0 x^i$;

(4) *Generalized Axiom of Revealed Preference* (GARP) if $x^i R x^j$ implies not $x^j P^0 x^i$.

The most common statement of the Strong Axiom is probably SARP 2.[4] It is clear that SARP 1 is equivalent to SARP 2. It is not quite so clear that SARP 3 is equivalent to SARP 2, but nevertheless they are equivalent. One can easily show that SARP 1, SARP 2, and SARP 3 imply GARP, but not vice versa. Basically SARP (in any of its formulations) requires single valued demand functions while GARP is compatible with multivalued demand functions. For example, the data in Figure 1 violate SARP but are quite compatible with GARP.

[3]One can always use the duality theorem of linear programming to construct an equivalent problem with n^2 variables and $2n$ constraints, but this problem may also be computationally difficult.

[4]See Richter **[22]** for several variations on revealed preference axioms. Note that Richter considers a framework where the entire demand correspondence is given, rather than only a finite number of observations. This leads to a number of differences in the analysis.

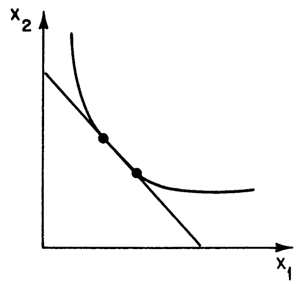

FIGURE 1.

This is why we refer to GARP as the *Generalized* Axiom of Revealed Preference. It turns out to be a necessary and sufficient condition for data to be consistent with utility maximization, and is in fact equivalent to Afriat's cyclical consistency condition.

FACT 1: *A set of data satisfies cyclical consistency if and only if it satisfies GARP.*

PROOF: Suppose that we have some data containing a violation of cyclical consistency so that $p^rx^r \geqq p^rx^s, \ldots, p^jx^j > p^jx^i, \ldots, p^qx^q \geqq p^qx^r$. Then x^iRx^j by going around the cycle, and $x^jP^0x^i$ directly. Hence we have a violation of GARP.

On the other hand, suppose we have some data that has a violation of GARP. Then writing out the violation in the above form shows we have a violation of cyclical consistency also.

The equivalency of GARP and cyclical consistency is trivial from the mathematical point of view, but is quite important from the computational point of view, since GARP is quite simple to check in practice, as we discuss below.

First, let us note that GARP can be restated as: if x^iRx^j then $p^jx^j \leqq p^jx^i$ for $i, j = 1, \ldots, n$. Hence verifying that some data satisfies GARP is trivial once we know the relation R—the transitive closure of the direct revealed preference relation R^0.

It is clear that the computation of the transitive closure of a finite relation is a finite problem. The only issue is how one might compute it efficiently. This question has been addressed in the economics literature by Koo [**14, 15, 16**], Dobell [7], and Uebe [**28**], and in the computer science literature by Warshall [**31**] and Munroe [**20**], among others.

Most of the algorithms in the economics literature compute the transitive closure of a relation in time proportional to n^4. The computer scientists, utilizing the law of comparative advantage, do a bit better. Warshall's algorithm computes the transitive closure in n^3 steps, and Munroe describes a process that does it in time proportional to $n^{2.74}$. Warshall's algorithm is especially easy to implement and quite ingenious. It seems fast enough for the problems encountered in economics, as well. We therefore describe Warshall's algorithm in Appendix 2.

At this point it might be worthwhile to be rather explicit about how one represents the relations R^0 and R in a form suitable for computation and how one actually verifies GARP in a systematic way.

Let us construct an n by n matrix M whose $i-j$ entry is given by:

$$m_{ij} = \begin{cases} 1 & \text{if } p^i x^i \geqq p^i x^j, \text{ that is, } x^i R^0 x^j; \\ 0 & \text{otherwise.} \end{cases}$$

M is constructed directly from the data; it summarizes the relation R^0. Warshall's algorithm, described in Appendix 2, operates on M to create a matrix MT where

$$mt_{ij} = \begin{cases} 1 & \text{if } x^i R x^j, \\ 0 & \text{otherwise.} \end{cases}$$

MT can be used to check GARP in the following way.

ALGORITHM 1: *Checking data for consistency with GARP.*

Inputs: (p^i, x^i), $i = 1, \ldots, n$, and the matrix MT representing the relation R.

Outputs: whether the data satisfies GARP or not.

1. Is $mt_{ij} = 1$ and $p^j x^j > p^j x^i$ for some i and j? If so, we have a violation of GARP.

Algorithm 1 is easily implemented on a computer. According to Afriat's theorem and Fact 1 we can use Algorithm 1 to simply and directly test a finite amount of data with the utility maximization model. If some data satisfies GARP then there is a nice utility function that will rationalize the observed behavior. If the data contains a violation of GARP then there does not exist a nonsatiated utility function that will rationalize the data. Hence we have a straightforward and efficient way to check a finite amount of data for consistency with the neoclassical model of consumer behavior.

2. RECOVERABILITY—ORDINAL COMPARISONS OF CONSUMPTION BUNDLES

Let us turn now to a somewhat different issue, namely the recoverability question described in the introduction. The revealed preference relation R which we discussed in the previous section summarizes all of the preference information contained in the demand observations. Any complete preference ordering that rationalizes the data must contain R, and every completion of R that rationalizes the data is a possible preference ordering that generated the data.

However, economists typically assume certain regularity conditions on the allowable preference orderings. For example we might restrict ourselves to preference orderings representable by utility functions that are nonsatiated, monotonic, and concave. Afriat's theorem implies that we can always impose such restrictions with no loss of generality; and conversely, that it is impossible to detect violations of these restrictions with a finite amount of demand data.

Suppose then that we are given two new consumption bundles x^0 and x' that have not been previously observed. Suppose that every continuous, nonsatiated, concave, monotonic utility function $u(x)$ that was consistent with (p^i, x^i), $i = 1, \ldots, n$, implied that $u(x^0) > u(x')$. Then we might well be justified in concluding that x^0 was in fact preferred to x'.

Alternatively we could adopt the following viewpoint. Suppose that *every* price vector p^0 at which x^0 could be demanded—and that was consistent with the data (p^i, x^i), $i = 1, \ldots, n$—also implied that x^0 was revealed preferred to x'. Then certainly we could conclude x^0 would be preferred to x' by any consistent consumer. Let us consider this approach in a bit more detail.

First it is clear that if x^0 has already been observed—so we know the price at which x^0 is demanded—there is no problem in verifying whether $x^0 R x'$. Hence we concentrate on the case where x^0 has not previously been observed. In this case we do not know what price to associate with x^0 for purposes of the revealed preference comparison. However, we do know what the set of *possible* prices could be:

DEFINITION: Given any bundle x^0 not previously observed we define the set of prices that *support* x^0 by:

$$S(x^0) = \left\{ p^0 : (p^i, x^i),\ i = 0, \ldots, n, \text{ satisfies GARP and } p^0 x^0 = 1 \right\}.$$

This is simply the set of prices at which x^0 could be demanded and still be consistent with the previously observed behavior. (The requirement that $p^0 x^0 = 1$ is a convenient normalization.) We note that Afriat's theorem implies $S(x^0)$ is nonempty for all x^0—just let p^0 be the supporting price at x^0 of any concave utility function that rationalizes the data.

We can use the definition of GARP to provide a convenient description of $S(x^0)$:

FACT 2: *A price vector p^0 is in $S(x^0)$ if and only if it satisfies the following system of linear inequalities*:

(1) $$p^0 x^0 = 1,$$

(2) $$p^0 x^0 \leqq p^0 x^i \quad \textit{for all } x^i \textit{ such that } x^i R x^0,$$
$$p^0 x^0 < p^0 x^i \quad \textit{for all } x^i \textit{ such that } x^i P x^0.$$

PROOF: Follows immediately from the definition of GARP.

According to Fact 2, $S(x^0)$ is simply the solution set to a certain system of linear inequalities constructed from the data (p^i, x^i), $i = 1, \ldots, n$, and the relations R and P.

We can use $S(x)$ to describe the set of observations "revealed worse" than x^0 and "revealed preferred" to x' in the following way.

$$RW(x^0) = \{x : \text{for all } p^0 \text{ in } S(x^0), p^0x^0 \geqq p^0x^i \text{ for}$$

$$\text{some } x^iPx \text{ or } p^0x^0 > p^0x^i \text{ for some } x^iRx\},$$

$$RP(x') = \{x : \text{for all } p \text{ in } S(x), px \geqq px^i \text{ for some}$$

$$x^iPx' \text{ or } px > px^i \text{ for some } x^iRx'\}.$$

More succinctly, and with only a slight abuse of our earlier definitions, we might write:

$$RW(x^0) = \{x : \text{for all } p^0 \text{ in } S(x^0), x^0Px\},$$

$$RP(x') = \{x : \text{for all } p \text{ in } S(x), xPx'\}.$$

These definitions formalize the idea described earlier: if x' is in $RW(x^0)$, then whatever the price at which x^0 is demanded—as long as it is consistent with the previous data—that price will necessarily make x^0 revealed preferred to x'. Thus *every* concave monotonic utility function that rationalizes the data must rank x^0 ahead of x'. Of course $RP(x')$ has a similar interpretation. In fact it is clear from the definitions that x^0 is "revealed preferred" to x' if and only if x' is "revealed worse" than x^0. We record this fact for future reference.

FACT 3: *x^0 is in $RP(x')$ if and only if x' is in $RW(x^0)$.*

$RP(x^0)$ and $RW(x^0)$ are extremely important to the rest of our discussion so it is worthwhile presenting a few two-dimensional examples. The simplest case—with one data point—is presented in Figure 2. Let us verify that Figure 2 is correct.

First, we consider $RP(x^0)$. In this simple case, $RP(x^0)$ is simply the convex monotonic hull of all points revealed preferred to x^0: namely x^1 and x^0 itself. To verify this, let x be any point in $RP(x^0)$, and let p be any (nonnegative) price vector at which x could be demanded. It is geometrically clear that, whatever budget line is chosen, x will be revealed preferred to x^0—either directly, or indirectly through the observation x^1. (The reader might check his understanding of this point by indicating the region where x will be *directly* revealed preferred to x^0 by all supporting prices, and the region where x will only be *indirectly* revealed preferred to x^0 for some supporting prices.) So much for $RP(x^0)$.

In order to verify the construction of $RW(x^0)$, we have to consider all of the prices at which x^0 could be demanded and still be consistent with the previous

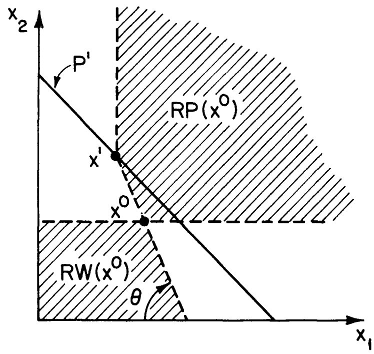

FIGURE 2.

data point (p^1, x^1). In this case GARP imposes an important restriction on p^0: the budget line through x^0 can be no steeper than the indicated angle θ. If it were steeper we would create a violation of GARP: we would have $x^1 R x^0$, and $x^0 P^0 x^1$. $RW(x^0)$ is the set of points that lie below *all* budget lines consistent with GARP —exactly as illustrated in Figure 2.

Figure 3 presents a more complex example. As before $RP(x^0)$ turns out to be the convex monotonic hull of all the points revealed preferred to x^0. $RW(x^0)$ is a bit more interesting. For all budgets that support x^0 and satisfy GARP, x^0 is revealed preferred to x^1, and *a fortiori* to all the points beneath x^1's budget set . . . including x^2, x^3 and so on.

Now Figure 3 presents us with quite a bit of information about the indifference curve passing through x^0: it cannot intersect $RP(x^0)$ or $RW(x^0)$—hence it must lie in between the two. Put another way, *the set of bundles preferred to* x^0 (using the true utility function) *must always contain* $RP(x^0)$, *and must be*

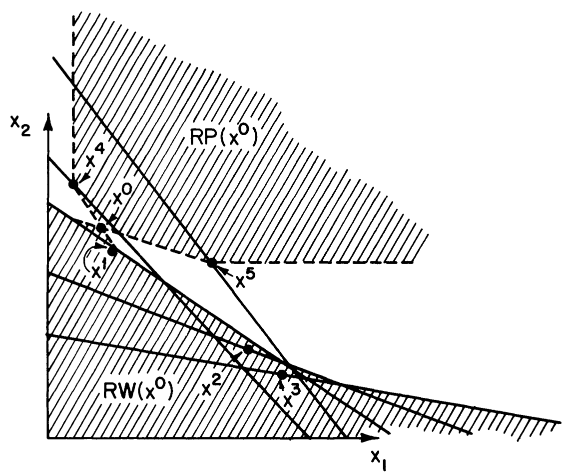

FIGURE 3.

contained in the complement of $RW(x^0)$. This last set, the complement of $RW(x^0)$, will be useful later on; we will call it $NRW(x^0)$ for "not revealed worse" than x^0.

It is clear from Figure 3 that $RP(x^0)$ and $NRW(x^0)$ are not only "inner" and "outer" estimates of the set of bundles preferred to x^0, they are also the tightest inner and outer estimates. If a point x' is not contained in either of these sets then there is a nice utility function that rationalizes the data for which $u(x^0) \geqq u(x')$ . . . and there is a nice utility function that rationalizes the data for which $u(x') \geqq u(x^0)$.

These statements are obvious for the two dimensional example given in Figure 3, but in fact they are true in general. In order to establish this we need the following criterion for membership in $RW(x^0)$.

FACT 4: *A bundle* x' *is in* $RW(x^0)$ *if and only if there does not exist a* $p^0 \geqq 0$ *that satisfies the following system of linear inequalities*:

$$(1) \qquad p^0x^0 = 1,$$

$$(2) \qquad \begin{aligned} & p^0x^0 \leqq p^0x^i && \textit{for all } x^i \textit{ such that } x^iRx^0, \\ & p^0x^0 < p^0x^i && \textit{for all } x^i \textit{ such that } x^iPx^0, \end{aligned}$$

$$(3) \qquad \begin{aligned} & p^0x^0 \leqq p^0x^j && \textit{for all } x^j \textit{ such that } x^jRx', \\ & p^0x^0 < p^0x^j && \textit{for all } x^j \textit{ such that } x^jPx'. \end{aligned}$$

PROOF: Suppose x' is in $RW(x^0)$. Then any p^0 that satisfies the first set of inequalities is a supporting price for x^0 by Fact 2. By the definition of $RW(x^0)$ it must therefore violate one of the inequalities in the second set.

Conversely suppose x' is not in $RW(x^0)$. Then there is some supporting price p^0 at which x^0 is not revealed preferred to x' by any chain. That is, p^0 satisfies (2) and (3).

Fact 4 gives us an explicit way to check whether x' is revealed worse than x^0. And by Fact 3 we can see whether x' is revealed preferred to x^0 just by checking whether x^0 is revealed worse than x'. Hence we can recover all of the ordinal information in the data by checking whether there exists a solution to a simple set of linear inequalities. This is easily accomplished by solving a simple linear program. Note that the number of constraints in this program will at most be $2n + 1$—and generally be considerably smaller than $2n + 1$.

We can now verify the intuitively plausible statements made earlier concerning the relationship between $RP(x^0)$, $RW(x^0)$, $P(x^0) = \{x : u(x) > u(x^0)\}$, and $W(x^0) = \{x : u(x^0) > u(x)\}$.

FACT 5: *Let* $u(x)$ *be any utility function that rationalizes the data. Then for all* x^0, $RP(x^0) \subset P(x^0) \subset NRW(x^0)$.

PROOF: Obvious from the fact that x^0Px' implies $u(x^0) > u(x')$ for any utility function that rationalizes the data.

FACT 6: *Suppose that x' is not in $RW(x^0)$; then there exists a nonsatiated, continuous, concave monotonic utility function that rationalizes the data for which $u(x^0) \geqq u(x')$. An analogous statement holds if x' is not in $RP(x^0)$.*

PROOF: Suppose x' is not in $RW(x^0)$. Then by Fact 4 there exists a p^0 supporting x^0 such that not $x^0 Px'$. Hence by using Fact 16 in Appendix 1, there is a utility function with the stated properties.

FACT 7: *Let $x^0 Rx'$. Then $RP(x^0) \subset RP(x')$. Assume further that x' is observed as a chosen bundle at some price p'. Then $RW(x^0) \supset RW(x')$ and $NRW(x^0) \subset NRW(x')$.*

PROOF: Let $\hat{x}$ be in $RP(x')$. Then for all $\hat{p}$ that support $\hat{x}$ we have $\hat{x}Rx^0$. Since by hypothesis $x^0 Rx'$, transitivity implies $\hat{x}Rx'$. Hence $\hat{x}$ is in $RP(x')$.

Let $\hat{x}$ be in $RW(x')$. Since x' is actually chosen at price p' this implies $x'R\hat{x}$. Since by hypothesis $x^0 Rx'$, transitivity implies $x^0 R\hat{x}$. Hence $\hat{x}$ is in $RW(x^0)$.

3. RECOVERABILITY—ORDINAL COMPARISONS OF BUDGETS

In many applications of demand analysis the natural objects of interest are not bundles of goods but are budgets—i.e. prices and expenditures. For example, if one wants to compare proposed changes in the tax structure, it is natural to compare alternative price configurations: given two proposed lists of prices and expenditures (p^0, y^0) and (p', y') we want to know which one is preferred by some individual consumer.

If we had a measure of the consumer's indirect utility function $v(p, y)$ we could simply compute $v(p^0, y^0)$ and $v(p', y')$ and compare the two numbers. If we have only a finite number of observations on a consumer's behavior (p^i, x^i), $i = 1, \ldots, n$, we could postulate a specification of an indirect utility function, derive the associated demand functions, and estimate the parameters of the resulting demand system. These estimated parameters of the demand system translate directly back to parameters of the indirect utility function which can then be used to make the welfare comparison between the two budgets.

However, the parametric specification necessarily involves an unwarranted maintained hypothesis of functional form. How can we proceed to make a nonparametric comparison of (p^0, y^0) versus (p', y')?

Let us recall the notion of *indirect revealed preference* of Sakai **[23]**, Little **[18]**, and Richter **[22]**.

DEFINITION: Given an observed budget (p^i, y^i) and a budget (p, y), we say:

(1) (p, y) is *directly revealed preferred* to (p^i, y^i), written $(p, y)R^0(p^i, y^i)$, if $px^i \leqq y$.

(2) (p, y) is *strictly directly revealed preferred* to (p^i, y^i), written $(p, y)P^0 (p^i, y^i)$, if $px^i < y$.

(3) (p, y) is *revealed preferred* to (p^i, y^i), written $(p, y)R(p^i, y^i)$, if R is the transitive closure of R^0.

(4) (p, y) is *strictly revealed preferred* to (p^i, y^i), written $(p, y)P(p^i, y^i)$ if there exist observed budgets (p^j, y^j) and (p^l, y^l) such that $(p, y)R(p^j, y^j)$, $(p^j, y^j)P(p^l, y^l)$, $(p^l, y^l)R(p, y)$.

Note that the indirect revealed preference relation works exactly opposite to the way the revealed preference relation works. To tell whether x^0 is revealed preferred to something we need to know the price p^0 at which x^0 is demanded—and then x^0 is revealed preferred to the infinite number of bundles beneath its budget line. To tell whether (p^0, y^0) is revealed worse than some budget we need to know the bundle x^0 that is demanded at (p^0, y^0)—and then (p^0, y^0) is revealed worse than the infinite number of budgets (p, y) for which $px^0 \leqq y$.

Nevertheless we can apply the same approach to ordinal comparisons to construct dual versions of the results in Section 3. This duality is most clearly exhibited if we normalize prices by dividing through by expenditure so that budgets are uniquely described by $p^0 = (p^0, 1)$ and $p' = (p', 1)$.

Definition: Given any price p^0 not previously observed we define the set of bundles that *support* p^0 by:

$$S(p^0) = \left\{ x^0 : (p^i, x^i),\ i = 0, \ldots, n, \text{ satisfies GARP and } p^0x^0 = 1 \right\}.$$

As before the requirement that $p^0x^0 = 1$ is only a normalization.

We can now describe the set of budgets "revealed preferred" or "revealed worse" than a given budget by:

$$RW(p^0) = \left\{ p : \text{for all } x \text{ in } S(p),\ 1 \geqq p^0x^i \text{ for some } p^iPp, \right.$$
$$\left. \text{or } 1 > p^0x^i \text{ for some } p^iRp \right\},$$
$$RP(p') = \left\{ p : \text{for all } x' \text{ in } S(p'),\ 1 \geqq px^i \text{ for some } p^iPp' \right.$$
$$\left. \text{or } 1 > px^i \text{ for some } p^iRx' \right\}.$$

Of course these definitions could also be stated as:

$$RW(p^0) = \left\{ p : \text{for all } x \text{ in } S(p),\ 1 \geqq p^0x^i \text{ for some } x^iPx, \right.$$
$$\left. \text{or } 1 > p^0x^i \text{ for some } x^iRx \right\},$$
$$RP(p') = \left\{ p : \text{for all } x' \text{ in } S(p'),\ 1 \geqq px^i \text{ for some } x^iPx' \right.$$
$$\left. \text{or } 1 > px^i \text{ for some } x^iRx' \right\}.$$

Or even more succinctly:

$$RW(p^0) = \{p : \text{for some } x \text{ in } S(p), p^0Rp\},$$

$$RP(p') = \{p : \text{for some } x' \text{ in } S(p'), pPp'\}.$$

We can now state the dual versions of Facts 2 and 4. The proofs are completely analogous and are left to the reader.

FACT 8: *A bundle x^0 is in $S(p^0)$ if and only if it satisfies the following system of linear inequalities*:

(1) $p^0x^0 = 1,$

(2) $p^ix^i \leqq p^ix^0$ *for all p^i such that p^0Rp^i*,
$p^ix^i < p^ix^0$ *for all p^i such that p^0Pp^i*.

FACT 9: *A budget p' is in $RP(p^0)$ if and only if there does not exist an $x^0 \geqq 0$ that satisfies the following system of linear inequalities*:

(1) $p^0x^0 = 1,$

(2) $p^ix^i \leqq p^ix^0$ *for all p^i such that p^0Rp^i*,
$p^ix^i < p^ix^0$ *for all p^i such that p^0Pp^i*,

(3) $p^jx^j \leqq p^jx^0$ *for all p^j such that $p'Rp^j$*,
$p^jx^j < p^jx^0$ *for all p^j such that $p'Pp^j$*.

Of course the dual versions of Facts 3, 5, and 6 are also true. The statement and proofs of these are left to the reader as well.

Another type of comparison that is often useful is to be able to compare bundles with budgets and vice versa. For example if we are given a direct and an associated normalized indirect utility function, $u(x)$ and $v(p)$, we could consider:

(1) All budgets p preferred to a bundle x^0:

$$PP(x^0) = \{p : v(p) > u(x^0)\}.$$

(2) All budgets p worse than a bundle x^0:

$$PW(x^0) = \{p : v(p) < u(x^0)\}.$$

(3) All bundles x preferred to a budget p^0:

$$XP(p^0) = \{x : u(x) > v(p^0)\}.$$

(4) All bundles x worse than a budget p^0:

$$XW(p^0) = \{x : u(x) < v(p^0)\}.$$

Each of these constructs has its "revealed preferred" and "revealed worse" analogy:

(1) All budgets p revealed preferred to a bundle x^0:

$$PRP(x^0) = \{p : \text{for all } x \text{ in } S(p),\ xPx^0\}.$$

(2) All budgets p revealed worse than a bundle x^0:

$$PRW(x^0) = \{p : \text{for all } p^0 \text{ in } S(x^0), \text{ and all } x \text{ in } S(p),\ x^0Px\}.$$

(3) All bundles x revealed preferred to a budget p^0:

$$XRP(p^0) = \{x : \text{for all } p \text{ in } S(x), \text{ and all } x^0 \text{ in } S(p^0),\ xPx^0\}.$$

(4) All bundles x revealed worse than a budget p^0:

$$XRW(p^0) = \{x : \text{for all } x^0 \text{ in } S(p^0),\ x^0Px\}.$$

If we want to verify whether p' is in $PRP(x^0)$, etc. we simply have to write down the associated system of linear inequalities following the general model of Facts 2 and 4. In cases (2) and (4) above, these systems involve unknown p's and unknown x's and are therefore somewhat involved. Cases (1) and (4) on the other hand are rather simple. We record this fact for future reference.

FACT 10:

$$PRP(x^0) = \{p : 1 > px^i \text{ for some } x^iRx^0 \text{ or } 1 \geqq px^i \text{ for some } x^iPx^0\},$$

$$XRW(p^0) = \{x : 1 > p^0x^i \text{ for some } x^iRx \text{ or } 1 \geqq px^i \text{ for some } x^iPx\}.$$

4. EXTRAPOLATION—FORECASTING DEMANDED BUNDLES

Suppose that we have observed choices (p^i, x^i), $i = 1, \ldots, n$, and that we are given some new budget $(p^0, 1)$ which has not been previously observed. What choice will the consumer make if his choice is to be consistent with the preferences revealed by his previous behavior? What is the best "overestimate" of the demanded bundle at p^0?

It turns out that we have already answered this question: it is simply the set of bundles that support the budget p^0, namely $S(p^0)$. For $S(p^0)$ is by definition all of the bundles of goods x^0 which make the data (p^i, x^i), $i = 0, \ldots, n$, consistent with GARP. It is therefore the tightest overestimate of the demand correspondence at p^0: every bundle in $S(p^0)$ *could* be a chosen bundle at p^0 and any

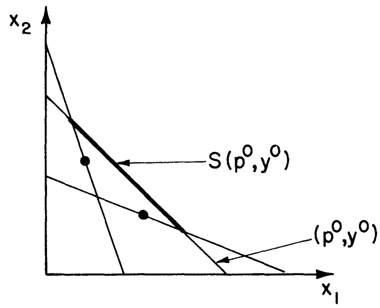

FIGURE 4.

bundle outside of $S(p^0)$ could never be chosen. Figure 4 gives a simple example of $S(p^0)$.

In an analogous manner $S(x^0)$ gives us the tightest overestimate of the *inverse* demand correspondence.

5. RECOVERABILITY—BOUNDING A SPECIFIC UTILITY FUNCTION

It is often desirable to know not only whether some bundle is preferred to some other bundle, but by *how much* one bundle is preferred to another. Now of course, there is no unique answer to this question: demand theory is completely ordinal in nature and there is no unique cardinal representation of utility. On the other hand it is a common practice to use certain specific cardinalizations of utility in measuring economic welfare.

One particularly useful cardinalization is Samuelson's "money metric" utility function (Samuelson [**25**]). For reasons that will become apparent, I prefer to call this function the *direct income compensation function*. We can define it in two equivalent ways:

$$m(p, x^0) = \inf px \quad \text{such that } x \text{ is in } P(x^0)$$

where $P(x^0) = \{x : u(x) > u(x^0)\}$ or,

$$m(p, x^0) = e(p, u(x^0)).$$

In the latter definition $e(p, u)$ is the expenditure function and $u(x)$ is the associated utility function. It is obvious from this latter definition that $m(p, x^0)$ behaves like an expenditure function with respect to p. It is also straightforward to show that for fixed p, $m(p, x^0)$ behaves like a *utility* function with respect to x^0: since the expenditure function is always increasing in utility, $m(p, x^0)$ is a monotonic transformation of a utility function and is therefore itself a utility function.

The direct income compensation function can be used to describe at least two measures of "how much" one configuration (p^0, x^0) is preferred to another configuration (p', x'), namely Hicks' compensating and equivalent variations:

$$C = m(p', x') - m(p', x^0),$$

$$E = m(p^0, x') - m(p^0, x^0).$$

Since $m(p^0, x)$ and $m(p', x)$ are each utility functions that represent the same preferences, C and E must always have the same sign, but they generally will have different magnitudes.

Let us accept for the moment that $m(p, x)$ is a reasonable cardinalization of utility. The question that then arises is how we might measure it. If we are given a parametric form for the utility function or expenditure function it is always possible to compute $m(p, x)$ directly. However, in the spirit of the nonparametric approach to demand analysis we ask how we might compare functions that provide bounds on $m(p, x)$ that are consistent with a finite set of observed demands (p^i, x^i), $i = 1, \ldots, n$.

In Section 2 we described the best inner and outer approximations to $P(x^0)$. It is natural to define the upper and lower bounds on the compensation function by:

$$m^+(p, x^0) = \inf px \quad \text{such that } x \text{ is in } RP(x^0),$$

$$m^-(p, x^0) = \inf px \quad \text{such that } x \text{ is in } NRW(x^0).$$

I refer to these as the *overcompensation* and the *undercompensation* functions respectively.

FACT 11: *Let m^+ and m^- be defined as above. Then*

(i) $m^+(p^0, x) \geqq m(p^0, x) \geqq m^-(p^0, x)$ *for all* p^0, x.

(ii) $x^i R x$ *implies* $m^+(p^0, x^i) \geqq m^+(p^0, x)$. *If* $x^i R x^j$ *and* x^j *is chosen at some price* p^j, *then* $m^-(p^0, x^i) \geqq m^-(p^0, x^j)$.

PROOF: (i) Follows from Fact 5. (ii) Follows from Fact 7.

Fact 11 shows that: (i) $m^+(p, x)$ and $m^-(p, x)$ do bound the compensation function, and (ii) they are themselves utility functions that respect the revealed preference ordering.

Thus the overcompensation and undercompensation functions provide theoretically ideal bounds to the compensation function. The problem with these two functions is that they are rather difficult to compute in practice. Recall that Fact 4 gave us a way to verify whether any given bundle x was an element of $RP(x^0)$ or $RW(x^0)$. However, I do not currently have any *explicit* description of these two sets of the sort suitable for mathematical programming techniques. So instead I have proceeded by defining two approximations to the overcompensation and undercompensation functions. These two approximations do provide bounds, but they are just not the theoretically tightest bounds. We turn now to a description of these approximations.

Let us define the convex, monotonic hull of $\{x^i : x^iRx^0\}$:

$$CM(x^0) = \text{interior of convex hull of } \{x : x \geqq x^i, x^iRx^0\}.$$

FACT 12: *$RP(x^0) \supset CM(x^0)$ for all x^0.*

PROOF: Let x be a point in $CM(x^0)$ and let p be any price vector that supports x. Then I claim $px > px^i$ for some x^iRx^0. For if not, p would separate x from $CM(x^0)$, a contradiction. Since xRx^i, x^iRx^0 we have that x is in $RP(x^0)$.

Then we can define the *approximate overcompensation function* by:

$$am^+(p, x^0) = \inf px \quad \text{such that } x \text{ is in } CM(x^0).$$

Since $CM(x^0)$ is a convex polytope whose vertices are precisely those x^iRx^0, we can also describe this minimization problem by:

$$am^+(p, x^0) = \min px^i, \qquad \text{such that } x^iRx^0.$$

Note that this function is quite simple to compute. Nevertheless, this approximate overcompensation function does share some desirable properties with the true overcompensation function.

FACT 13:

(1) $am^+(p, x) \geqq m^+(p, x) \geqq m(p, x)$.

(2) *x^0Rx implies $am^+(p, x^0) \geqq am^+(p, x)$.*

(3) *There exists a convex monotonic preference order $\succsim$ such that*

$$am^+(p, x^0) = m(p, x^0) \text{ for all } x^0.$$

PROOF: The first two parts are obvious. The third is rather detailed. First we define the order and verify that it works; then we establish its properties.

Let $x \gtrsim x'$ if and only if $am^+(p,x) \geqq am^+(p,x')$. Let us show that the compensation function that goes along with this order is in fact equal to $am^+(p,x)$.

Let px^* solve:

$$px^* = m(p,\bar{x}) = \min px \quad \text{such that } am^+(p,x) \geqq am^+(p,\bar{x})$$

and let $p\tilde{x}$ solve

$$p\tilde{x} = am^+(p,\bar{x}) = \min px^i, \qquad \text{such that } x^iR\bar{x}.$$

Now $\tilde{x}R\bar{x}$ so property (2) shows that $am^+(p,\tilde{x}) \geqq am^+(p,\bar{x})$. Hence $\tilde{x}$ is feasible for the first problem and therefore $px^* \leqq p\tilde{x}$.

On the other hand

$$px^* \geqq am^+(p,x^*) \geqq am^+(p,\bar{x}) = p\tilde{x}.$$

Next we examine the properties of the preference ordering $\gtrsim$.

(a) $\{x : am^+(p,x) \geqq k\}$ is convex. To prove this, we suppose $am^+(p,x') \geqq k$ and $am^+(p,x'') \geqq k$. Let

$$A = \{x^i : x^iRx'\},$$

$$B = \{x^i : x^iRx''\},$$

$$C = \{x^i : x^iR(tx' + (1-t)x'')\} \qquad \text{for some } t \text{ such that } 0 \leqq t \leqq 1.$$

I claim that if x^i is in C, then x^i is in $A \cup B$. For to say x^i is in C is to say that there exists a finite sequence such that:

$$p^ix^i \geqq p^ix^r,$$

$$p^rx^r \geqq p^rx^s,$$

$$p^lx^l \geqq p^l(tx' + (1-t)x'').$$

From the last inequality it is easy to show that either $p^lx^l \geqq p^lx'$ or $p^lx^l \geqq p^lx''$, which establishes the claim.

Now, since $C \subset A \cup B$, we have:

$$k \leqq \min_{x \text{ in } A \cup B} px \leqq \min_{x \text{ in } C} px = am^+(p, tx' + (1-t)x'').$$

(b) If $x' \geqq x^0$, then $am^+(p,x') \geqq am^+(p,x^0)$. This follows since $\{x^i : x^iRx'\} \subset \{x^i : x^iRx^0\}$.

Thus $am^+(p^0,x)$ is a utility function that bounds the compensation function and the bound is uniformly tight in the sense that there exists a "nice" preference ordering that actually generates $am^+(p^0,x)$ as its compensation function.

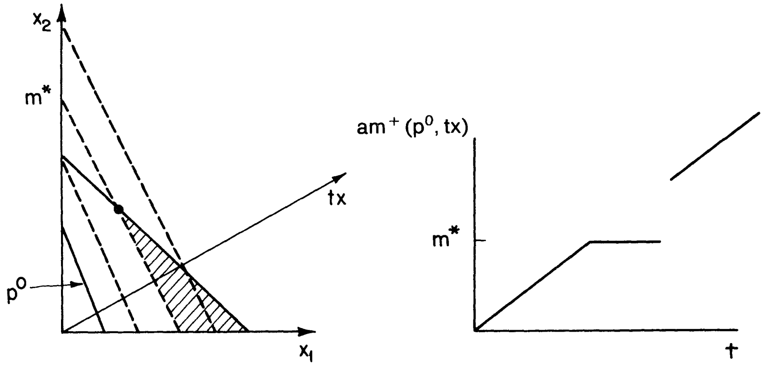

FIGURE 5.

However it must be pointed out that this ordering typically exhibits regions of satiation, and is in general discontinuous. An example is given in Figure 5. Here all the points in the shaded region are assigned $am^+(p^0, x) = p^0x^1$. The approximate overcompensation function increases linearly as one moves out the ray tx, then is constant, and then jumps discontinuously.

We turn now to the problem of computing an approximation to the undercompensation function. The basic trick here is to get an "inner bound" to $RW(x^0)$ by eliminating the nonconvexities shown in Figure 3. We define this inner bound by:

$$IRW(x_0) = \{x : \text{for all } p^0 \text{ in } S(x^0), x^0Rx^i, x^i \neq x^0 \text{ and } p^ix^i \geqq p^ix\}.$$

The crucial difference between $RW(x^0)$ and $IRW(x^0)$ is the requirement that $x^i \neq x^0$. This is made clear in Figure 3. The complement of $IRW(x^0)$, $NIRW(x^0)$, is then given by:

$$NIRW(x^0) = \{x : p^ix > p^ix^i \text{ for some } x^i \neq x^0 \text{ such that } x^0Rx^i$$
$$\text{for all } p^0 \text{ in } S(x^0)\}.$$

This is simply a set of a points defined by a finite number of linear inequalities. Hence there is no problem in computing the "approximate undercompensation function":

$$am^-(p^0, \bar{x}) = \inf p^0x$$
$$\text{such that } x \text{ is in } NIRW(\bar{x}).$$

This also shares some desirable features with the true undercompensation function:

FACT 14:

(1) $m(p,x) \geqq m^-(p,x) \geqq am^-(p,x)$,

(2) x^0Rx^j *implies* $am^-(p,x^0) \geqq am^-(p,x^j)$.

PROOF: Left to the reader.

Thus $am^-(p,x)$ bounds the true undercompensation function and it respects the revealed preference ordering, although it does not provide the theoretically ideal bound.

6. RECOVERABILITY—BOUNDING A SPECIFIC INDIRECT UTILITY FUNCTION

It is natural to extend the results of the last section to indirect utility comparisons. The function one wishes to bound is the *indirect income compensation function*

$$\mu(q;\, p,\, y) \equiv e(q, v(p, y))$$

where $e(q,u)$ is the expenditure function and $v(p,y)$ is the indirect utility function.[5] An equivalent way to define $\mu(q;\, p,\, y)$ is:

$$\mu(q;\, p,\, y) = \inf qx \quad \text{such that } x \text{ is in } XP(p,\, y) = \{x : u(x) > v(p,\, y)\}.$$

Applying the approach of the last section, it appears natural to define the *indirect overcompensation function* and the *indirect undercompensation function* by:

$$\mu^+(q;\, p,\, y) = \inf qx \quad \text{such that } x \text{ is in } XRP(p,\, y),$$

$$\mu^-(q;\, p,\, y) = \inf qx \quad \text{such that } x \text{ is in } NXRW(p,\, y).$$

Recall that $XRP(p,\, y)$ consists of all bundles revealed preferred to the budget $(p,\, y)$, and $NXRW(p,\, y)$ consists of all bundles not revealed worse than the budget $(p,\, y)$; formal definitions were given in Section 3.

It is by now straightforward to verify the following fact:

[5] The indirect compensation function was first discussed by McKenzie [**19**]. It has been extensively treated by Hurwicz and Uzawa [**13**].

FACT 15: *The indirect over and under compensation functions have the following properties*:

(i) $\mu^-(q; p, y) \leqq \mu(q; p, y) \leqq \mu^+(q; p, y)$.

(ii) $(p^0, y^0)R(p', y')$ *implies* $\mu^-(q; p^0, y^0) \geqq \mu^-(q; p', y')$.

If (p^0, y^0) *is the budget for some observed choice then* $\mu^+(q; p^0, y^0) \geqq \mu^+(q; p', y')$.

Let us now consider the computability of μ^+ and μ^-. As before, we can verify whether any given x' is an element of $XRP(p, y)$ by solving a set of linear inequalities; however it seems difficult to get an explicit description of the sort necessary for mathematical programming.

I therefore suggest the following approximation to μ^+:

$$a\mu^+(q; p, y) = am^+(q, x^i) \quad \text{if } (p, y) = (p^i, y^i) \text{ for some observed } (p^i, y^i),$$
$$= \max qx \quad \text{such that } x \text{ is in } S(p, y) \text{ otherwise.}$$

That is, if (p, y) is observed, we use the value of the approximate overcompensation function. Otherwise, we adopt the most conservative estimate and set $a\mu^+(q; p, y)$ equal to the maximum expenditure over all bundles in the "overestimate" of the demand correspondence. This clearly gives an upper bound on the true overcompensation function.

The indirect undercompensation function is, on the other hand, quite simple to compute. Since Fact 10 gives an explicit description of $XRW(p, y)$, as the solution set to a system of linear inequalities, we can simply compute $\mu^-(q; p, y)$ by solving a small linear program. An illustration of $XRW(p, y)$ and $\mu(q; p, y)$ is given in Figure 6.

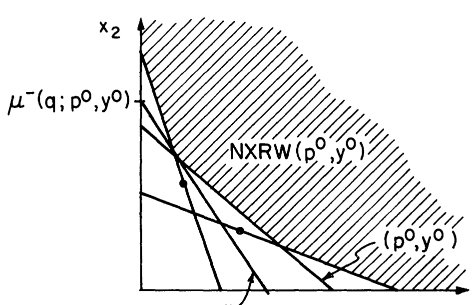

FIGURE 6.

7. SOME APPLICATIONS

The algorithms described in the previous sections have been assembled in a package of FORTRAN subroutines available from the author. Here I will briefly describe some computational experience with these routines.[6]

First let us consider the issue of testing demand data for consistency with preference maximization. I have applied the routines of Section 1 to several sets of aggregate consumption data. In each case the aggregate consumption data was consistent with GARP: that is, it could have been generated by a single neoclassical "representative consumer." At first glance this may seem somewhat surprising given the negative theoretical results of Sonnenschein [**27**] and Debreu [**8**]. However, upon reflection, it is not difficult to understand why this occurs.[7]

Most existing sets of aggregate consumption data are post-war data, and this period has been characterized by small changes in relative prices and large changes in income. Hence, each year has been revealed preferred to the previous years in the sense that it has typically been possible in a given year to purchase the consumption bundles of each of the previous years. Hence no "revealed preference" cycles can occur and the data are consistent with the maximization hypothesis. This observation implies that those studies which have rejected the preference maximization using conventional parametric techniques are rejecting only their particular choice of parametric form.

Given that a set of aggregate consumption data are consistent with preference maximization, we can compute the over- and undercompensation functions described in Sections 5 and 6. One can use these functions to provide some interesting bounds on cost of living indices.

Let (p^i, y^i) be a budget in year i and (p^0, y^0) be a budget in the base year. Then the *true cost of living index* is defined by:

$$i = \frac{\mu(p^0; p^i, y^i)}{y^0}.$$

The true cost of living index measures how much money one would need in the base year to be as well off as one was in the comparison year expressed as a fraction of base year expenditure. In order to calculate i one needs the indirect income compensation function which is equivalent to requiring complete knowledge of the individual preference ordering over some range.

However, we can use the results of Section 6 to compute upper and lower bounds on i that are consistent with any finite set of data. Table I presents the results of such a computation using U.S. aggregate consumption data by nine categories from 1947–78.

Note the tightness of the bounds. Typically the overestimate is within 15 per

[6]Diewert and Parkan [**10**] discuss their computational experience with some alternative nonparametric techniques.

[7]For another independent recent application of revealed preference methodology to aggregate data see Landsburg [**17**].

TABLE I
UPPER AND LOWER BOUND ON TRUE COST OF LIVING INDEX[a]
(CLASSICAL BOUNDS IN PARENTHESES)

Year	Upper Bound	Lower Bound
1947	.2496	.1841
1948	.2666	.2004
1949	.2715	.2024
1950	.2906	.2113
1951	.3107	.2237
1952	.3246	.2401
1953	.3409	.2548
1954	.3497	.2634
1955	.3744	.2886
1956	.3905	.3013
1957	.4096	.3172
1958	.4205	.3324
1959	.4500	.3596
1960	.4682	.3779
1961	.4806	.3903
1962	.5082	.4208
1963	.5342	.4499
1964	.5707	.4865
1965	.6119	.5342
1966	.6581	.5864
1967	.6906	.6089
1968	.7524	.6809
1969	.8089	.7406
1970	.8553	.8104
1971	.9174	.8906
1972	1.0000	1.0000
1973	1.0960	1.0409
1974	1.1900	1.0478 (0.9496)
1975	1.2994	1.0623 (1.0466)
1976	1.4354	1.1615
1977	1.5767	1.2764
1978	1.7330	1.4404

[a] Data are U.S. consumption data by 9 categories from the NBER Time Series Database (Tables 2.3 and 2.4). The goods are motor vehicles, furniture, other durables, food, clothing, gasoline and oil, housing, transportation, and other services.

cent of the underestimate which allows for a fairly tight estimate of the true cost of living. However, the accuracy of the table is slightly misleading in the following sense.

Given only the information contained in the two observations (p^0, y^0) and (p', y') it is possible to construct the classical bounds depicted in Figure 7. Improvements in these bounds are possible only when some budget set from another sample observation intersects the budget set given by (p', y') as in Figure 8.

Given the nature of the data, these intersections are quite rare, and in fact only occur for two years 1974 and 1975. Again, the lack of variation in the price data

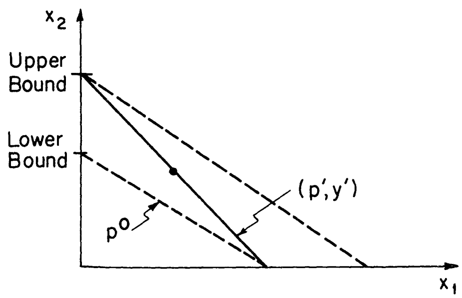

FIGURE 7.

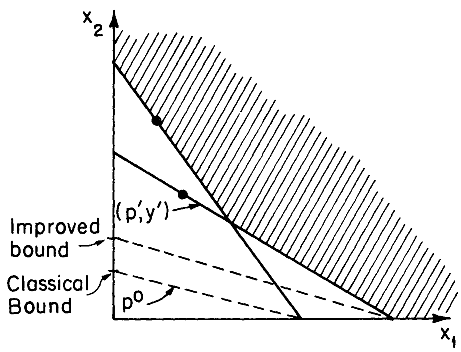

FIGURE 8.

limits the power of these methods in this case. However, the techniques proposed here do provide an improvement on the classical bounds when sufficient variation in price data is present.

8. SUMMARY

We have shown how the nonparametric techniques of revealed preference analysis can be used to: (1) test a finite amount of data for consistency with preference maximization model; (2) construct a nicely behaved utility function capable of rationalizing a finite amount of demand data; (3) compare previously unobserved consumption bundles and budgets with respect to their ordinal rankings; (4) compute cardinal bounds on the direct and indirect compensation functions; and (5) compute estimates of the direct and indirect demand correspondence consistent with previously observed demand data.

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APPENDIX I: A PROOF OF AFRIAT'S THEOREM

In this appendix we give a proof of Afriat's theorem. The proof we give is based on earlier proofs by Afriat [**4**] and Diewert [**9**], but is somewhat more constructive. In fact we will exhibit an algorithm which will actually compute a utility function which rationalizes any given finite amount of data. It turns out that it is convenient to first describe the algorithm to do this computation and then verify that it works in the course of the proof of Afriat's theorem.

The algorithm that we describe below makes use of a subroutine which calculates a maximal element of a finite set with respect to some binary relation.

Let us recall the following definition.

DEFINITION: An element x^m of a set S is *maximal* with respect to a binary relation B if x^iBx^m implies x^mBx^i.

If x^m is a maximal element then either there is nothing that is ranked ahead of it or the only things that are "ahead" of it are things that are indifferent to it.

If we have a finite set with a reflexive and transitive binary relation then there is always at least one maximal element; the following algorithm shows us how to find it. (See Sen [**26**, p. 11].)

ALGORITHM 2: *Finding a maximal element.*

Input: a reflexive and transitive binary relation B defined on a finite set $S = (x^1, \ldots, x^n)$ indexed by $I = (1, \ldots, n)$.

Output: an index m where x^iBx^m implies x^mBx^i.

1. Set $m = 1$, $b^0 = x^1$.
2. For each $i = 1, \ldots, n$, if x^iBb^{i-1} set $b^i = x^i$, and $m = i$. Otherwise set $b^i = b^{i-1}$.

We will let max(I) be a routine that performs Algorithm 2; that is, given a set S indexed by I, max(I) returns the index of a maximal element in S.

It is perhaps not immediately obvious that Algorithm 2 works. Hence we provide the following proof.

FACT 15: *The output of Algorithm* 2 *is the index of a maximal element of* S.

PROOF: First we note that by the transitivity and reflexivity of B, b^nBb^j for all $j = 0, \ldots, n$. Also note that $x^m = b^n$.

Now suppose we are given some x^iBx^m; i.e. x^iBb^n. We must show that b^nBx^i. First we observe that since x^iBb^n, and b^nBb^{i-1}, then x^iBb^{i-1}. Line 2 of the algorithm then implies $b^i = x^i$. But then b^nBb^i, $b^i = x^i$ gives b^nBx^i as required.

We note that the revealed preference relation R is transitive and reflexive, so Algorithm 2 will therefore correctly compute a maximal element. We can now present an algorithm which calculates numbers that satisfy the Afriat inequalities:

ALGORITHM 3: *Constructing the Afriat numbers.*

Input: A set of demand observations (p^i, x^i), $i = 1, \ldots, n$, and the revealed preference relation R that satisfy GARP.

Output: A set of numbers $U^i, \lambda^i > 0$, $i = 1, \ldots, n$, that satisfy the Afriat inequalities.

1. $I = \{1, \ldots, n\}$, $B = \emptyset$.
2. Let $m = \max(I)$.
3. Set $E = \{i \text{ in } I : x^iRx^m\}$. If $B = \emptyset$, set $U^m = \lambda^m = 1$ and go to 6. Otherwise go to 4.
4. Set $U^m = \min_{i \in E}\min_{j \in B}\min\{U^j + \lambda^j p^j(x^i - x^j), U^j\}$.
5. Set $\lambda^m = \max_{i \in E}\max_{j \in B}\max\{(U^j - U^m)/p^i(x^j - x^i), 1\}$.
6. Set $U^i = U^m$, $\lambda^i = \lambda^m$ for all $i \in E$.
7. Set $I = I \backslash E$, $B = B \cup E$. If $I = \emptyset$, stop. Otherwise, go to 2.

It is not at all obvious that Algorithm 3 does in fact compute numbers that satisfy the Afriat inequalities; however that fact will be verified in the proof of Afriat's theorem.

AFRIAT'S THEOREM: *The following conditions are equivalent*:

(1) *There exists a nonsatiated utility function that rationalizes the data.*

(2) *The data satisfies GARP*: *if* x^iRx^j, *then* $p^jx^j \leqq p^jx^i$.

(3) *There exist numbers* $U^i, \lambda^i > 0$ *such that* $U^i \leqq U^j + \lambda^j p^j(x^i - x^j)$ *for* $i, j = 1, \ldots, n$.

(4) *There exists a nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.*

PROOF: (1)$\Rightarrow$(2). Let $u(x)$ rationalize the data. If $p^ix^i \geqq p^ix^j$ then $u(x^i) \geqq u(x^j)$ by definition so that $x^iR^0x^j$ implies $u(x^i) \geqq u(x^j)$. If $p^ix^i > p^ix^j$ so that $x^iP^0x^j$, then I claim that $u(x^i) > u(x^j)$. If not, then $u(x^i) = u(x^j)$. But by local nonsatiation there is then an $\hat{x}$ such that $p^ix^i > p^i\hat{x}$ and $u(\hat{x}) > u(x^i)$. But then $u(x)$ could not rationalize the data point (p^i, x^i). Hence $x^iP^0x^j$ implies $u(x^i) > u(x^j)$, and GARP follows.

(2)$\Rightarrow$(3). In order to prove this we need to verify that Algorithm 3 works; i.e., that the numbers it calculates do indeed satisfy the Afriat inequalities.

At each pass through the algorithm a set of indices of "equivalent" elements, E, is removed from I and added to B, a set of indices of "better" elements. We will show that after step 6 is executed, the U's and the λ's at that stage satisfy the Afriat inequalities for all the U's and λ's calculated up to that point. That is, we will verify the following three statements:

(a) $$U^i \leqq U^j + \lambda^j p^j(x^i - x^j) \qquad \text{for all } j \text{ in } B \text{ and all } i \text{ in } E,$$

(b) $$U^j \leqq U^i + \lambda^i p^i(x^j - x^i) \qquad \text{for all } j \text{ in } B \text{ and all } i \text{ in } E,$$

(c) $$U^i \leqq U^j + \lambda^j p^j(x^i - x^j) \qquad \text{for all } i \text{ and } j \text{ in } E.$$

Proof of (*a*): By step 4 of the algorithm:

$$U^i = U^m \leqq U^j + \lambda^j p^j(x^i - x^j) \qquad \text{for all } j \text{ in } B \text{ and all } i \text{ in } E.$$

Proof of (*b*): First note that when the algorithm correctly executes statement 5, $p^i(x^j - x^i) > 0$, for all j in B. If not, x^iRx^j for some j in B. But then i would have been moved into B before j was moved into B.

Hence, the division is well defined and

$$\lambda^i = \lambda^m \geqq \frac{U^j - U^i}{p^i(x^j - x^i)} \qquad \text{for all } j \text{ in } B \text{ and all } i \text{ in } E.$$

Cross multiplying:

$$\lambda^i p^i(x^j - x^i) \geqq U^j - U^i \qquad \text{for all } j \text{ in } B \text{ and all } i \text{ in } E$$

which proves (b).

Proof of (*c*): First note that i, j in E implies $p^j(x^i - x^j) \geqq 0$. If not $x^jP^0x^i$, giving a violation of GARP. Now for all i and j in E:

$$U^i = U^j \qquad \text{and} \qquad \lambda^j = \lambda^m > 0$$

so

$$U^i \leqq U^j + \lambda^j p^j(x^i - x^j).$$

(3)$\Rightarrow$(4). We define the function $U(x)$ by

$$U(x) = \min_i \{ U^i + \lambda^i p^i(x - x^i) \}.$$

It is clear from the definition that this piecewise linear function has the stated properties. Hence we only need to verify that it rationalizes the data.

First we note that $U(x^i) = U^i$ for all $i = 1, \ldots, n$. For suppose the minimum is attained at x^m; then

$$U(x^i) = U^m + \lambda^m p^m(x^m - x^i) \leqq U^i$$

since $\lambda^m p^m(x^i - x^i) = 0$. But if this inequality were ever strict we would violate one of the Afriat inequalities.

Now suppose we are given some x such that $p^j x^j \geqq p^j x$. We must show that $U(x^j) \geqq U(x)$. This follows directly from the following set of inequalities:

$$\begin{aligned} U(x) &= \min_i \{ U^i + \lambda^i p^i (x - x^i) \} \\ &\leqq U^j + \lambda^j p^j (x - x^j) \\ &\leqq U^j = U(x^j) \end{aligned}$$

since $\lambda^j p^j (x - x^j) \leqq 0$.

(4)$\Rightarrow$(1). This is obvious.

It is worthwhile giving a somewhat more heuristic argument for Afriat's Theorem, which more directly exhibits the meaning of the Afriat inequalities. Suppose that we have a differentiable concave utility function that rationalizes some data (p^i, x^i), $i = 1, \ldots, n$. Then concavity implies

$$u(x^i) \leqq u(x^j) + Du(x^j)(x^i - x^j)$$

and utility maximization implies

$$Du(x^j) = \lambda^j p^j.$$

Putting these together we see that the Afriat conditions are a necessary condition for utility maximization in this differentiable framework. To motivate the sufficiency result we simply note that by concavity we have *n overestimates* of the utility at some point x since

$$u(x) \leqq u(x^i) + \lambda^i p^i (x - x^i) \qquad \text{for} \qquad i = 1, \ldots, n.$$

Hence the minimum of the right hand side over all observation i—the lower envelope—should give us a reasonable measure of the utility of x.

This interpretation of the U^i's as utility levels and the λ^i's as the marginal utilities of income was first suggested by Afriat [**1**] and further elucidated by Diewert and Parkan [**10**]. Varian [**29**, **30**], has used this sort of argument to derive finite necessary and sufficient conditions for a number of specializations of the utility maximization model.

Finally we give a proof of one last fact concerning Afriat's construction that was stated without proof at one point in the text. If x^i is not revealed preferred to x^j, then it is intuitively plausible that there is a nice utility function that rationalizes the data for which $u(x^j) \geqq u(x^i)$. This is verified in the next statement.

FACT 16: *If not $x^i R x^j$, then there is a nonsatiated, continuous, concave, monotonic utility function that rationalizes the data for which $u(x^j) \geqq u(x^i)$.*

PROOF: Simply ensure that max(I) returns the index j before the index i. Line 4 of Algorithm 3 then implies that $u(x^j) \geqq u(x^i)$.

APPENDIX II: COMPUTING THE TRANSITIVE CLOSURE

The following discussion concerning the computation of the transitive closure of a relation is taken from Aho and Ullman [**6**], which in turn is based on Warshall [**31**]. Their results are very slightly generalized in a way that is useful in some other applications (Varian [**29**]).

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

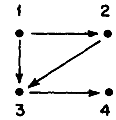

FIGURE 9.

Let M be an n by n matrix representing a binary relation; i.e. $m_{ij} = 1$ if $x^i R^0 x^j$ and $m_{ij} = 0$ otherwise. We can also think of M as representing a directed graph as in Figure 9: there is an arrow from vertex i to vertex j if and only if $m_{ij} = 1$. It is this interpretation that gives rise—somewhat indirectly—to Warshall's algorithm.

Suppose now that we have an arbitrary directed graph and some associated cost function c_{ij} where $c_{ij} \geqq 0$ measures the cost of transporting one unit of a good directly from vertex i to vertex j. If vertex i and vertex j are not directly connected c_{ij} is by definition infinite. Now although the cost of moving i to j *directly* is given by c_{ij}, the *cheapest* cost of moving i to j may be much less. Warshall's algorithm is concerned with calculating the least cost of moving from any vertex to any other vertex. We denote the magnitude of this least cost by $\bar{c}_{ij}$.

I claim that if we can solve this "least cost problem" we can easily solve the "transitive closure" problem. We just create a cost matrix C where

$$c_{ij} = \begin{cases} 1 & \text{if } m_{ij} = 1, \\ \infty & \text{if } m_{ij} = 0. \end{cases}$$

Now we run C through Warshall's algorithm to compute the least cost matrix $(\bar{c}_{ij})$. Then if $\bar{c}_{ij} = l < \infty$ we know that there is some path of length l that connects vertex i with vertex j. Hence a method to solve the least cost problem gives us a method to solve the transitive closure problem.

ALGORITHM 4: *Minimum cost of paths in a graph.*
Input: c_{ij} = cost of moving from node i to node j; $c_{ij} \geqq 0$.
Output: $\bar{c}_{ij}$ = *minimum* cost of moving from node i to node j.
(1) Set $k = 1$.
(2) For all i and j, if $c_{ij} \geqq c_{ik} + c_{kj}$ set $c_{ij} = c_{ik} + c_{kj}$.
(3) If $k < n$, let $k = k + 1$ and go to 2. If $k = n$, set $\bar{c}_{ij} = c_{ij}$ for all i and j.

It is not at all obvious that Algorithm 4 does indeed compute the minimum cost of moving from i to j for all i and j. But the following argument shows that it works.

FACT 17: *Let $(i, l, \ldots, m, j)$ be a path from i to j. Then* $\bar{c}_{ij} \leqq c_{il} + \cdots + c_{mj}$.

PROOF: Consider the algorithm when it has completed step (2). We will show that c_{ij} is the cost of the cheapest path from i to j that passes through no intermediate vertex with index greater than k. This is certainly true for $k = 1$, and we suppose it to be true for $k - 1$.

Let $(i, l, \ldots, m, j)$ be a path from i to j that passes through no intermediate vertex with index greater than k. If it does not pass through vertex k we are done. If it does pass through k, we can suppose it only passes through once, since removing a cycle cannot increase the cost. By the induction hypothesis c_{ik} is the cheapest path from i to k with no intermediate vertex greater than $k - 1$ and similarly for c_{kj}. Since step (2) of the algorithm ensures $c_{ij} \leqq c_{ik} + c_{kj}$, we are done.

Note that step (2) of the algorithm will be executed n^3 times; thus we can compute the transitive closure of a relation in n^3 computer additions and comparisons. Of course, if we are using Warshall's algorithm only to compute the transitive closure of a relation we can improve a bit on that bound. Consider for example the following FORTRAN subroutine which computes the transitive closure of a relation represented by the matrix M.

ALGORITHM 5: *Computing the transitive closure.*

Input: $M(I,J)=1$ if $p^ix^i \geqq p^ix^J$, 0 otherwise. N = number of observations; nobs = maximum number of observations.

Output: $M(I,J)=1$ if x^iRx^J, 0 otherwise.

```
      SUBROUTINE TCLSR (M, N)
      DIMENSION M(nobs, nobs)
      DO 30 K = 1, N
      DO 20 I = 1, N
      DO 10 J = 1, N
      IF (M(I, K) .EQ. 0 .OR. M(K, J) .EQ. 0) GO TO 10
      M(I, J) = 1
10    CONTINUE
20    CONTINUE
30    CONTINUE
      RETURN
      END
```

This clearly computes the transitive closure by a straightforward modification of the argument given in Fact 17.

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Testing Strictly Concave Rationality*

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We prove that the Strong Axiom of Revealed Preference tests the existence of a strictly quasiconcave (or strictly concave, strictly monotone) utility function generating finitely many demand observations. This sharpens earlier results of Afriat, Diewert, and Varian that tested ("nonparametrically") existence of a piecewise linear utility function For finite data sets, one implication of our result is that even some weak types of rational behavior—maximization of pseudotransitive or semitransitive preferences—are observationally equivalent to maximization of continuous, strictly concave, and strictly monotone utility functions. And for infinitely many observations, our result is the basis of several new rationality theorems. *Journal of Economic Literature* Classification Numbers 022, 213

1. INTRODUCTION

In applied economics, it is common to assume that consumers maximize continuous, strictly quasi concave, and monotone utility functions. Even strict concavity is often assumed. What restrictions do these special assumptions put on observable data? Can we test demand behavior to see whether it maximizes such a special function?

For a finite number of observations, we will give a complete answer (Theorem 1): Houthakker's Strong Axiom of Revealed Preference is a necessary and sufficient behavioral test for such "special rationality." This

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will imply that the special rationality is observationally equivalent to the much weaker hypothesis that the demand function maximizes some reflexive, transitive, and total preference: No finite set of data can distinguish between those assumptions. In fact, our results imply that finite sets of data cannot distinguish even much weaker types of rationality from the special rationality (Theorem 3).

Our proof is based on a Theorem of the Alternative. This not only yields a constructive proof, but also connects our result with others in revealed preference theory that employ similar tools. It allows the direct construction of continuous, strictly concave, strictly monotone utility-rationalizations (Theorem 2).

In Section 5 we relate our results to work by Afriat, Diewert, and Varian. That line of work—sometimes under the name of "nonparametric demand analysis"—tested for the existence of a piecewise linear utility, which could only rationalize demand in a much weaker sense. We also relate our work to the recent paper of Chiappori and Rochet on C^∞ rationalizations for the special case of invertible demand functions.

In Section 6 we note several applications of our theorems. They lead to the strengthening of many results—in approximation theory [11], in nontransitive consumer theory [10], in classical consumer integrability and revealed preference theory [8, 7, 17], and in the theory of equilibrium correspondences [2].

2. Rationality

We study demand data for n commodity types, where each bundle of commodities can be represented by a vector in some convex subset X of R^n. (Commonly X is assumed to be a subset of $R^n_{\geq}$, but we do not require that.)

We are interested in competitive consumers, so we denote by $B(p, m)$ the budget set determined by price vector $p = (p_1, ..., p_n) \in R^n_{>}$ and income $m \in R^l_{\geq}$:

$$B(p, m) = \{x \in X: p \cdot x \leqq m\}. \tag{2.1}$$

Often we write (p, m) for $B(p, m)$. We denote by $\mathscr{C}$ the family of all such budget sets. Sometimes we are interested in a particular subfamily $\mathscr{B} \subseteq \mathscr{C}$. For example, in this paper, we study choices from a finite collection $\mathscr{B}$ of budgets.

The set of bundles chosen under budget $B(p, m)$ will be denoted by $h(p, m)$. It is reasonable to assume that $h(p, m) \subseteq B(p, m)$. This defines then a correspondence h, which we call a *choice*. Of course, in the classical case of a demand *function*, h is singleton valued

Using the notions of Richter [15], we will call the choice h *rational* (on $(X, \mathscr{B})$), if there exists a binary relation $\succcurlyeq$ on X such that, for all $(p, m) \in \mathscr{B}$,

$$h(p, m) = \{x \in B(p, m): \forall y_{y \in B(p,m)}\ x \succcurlyeq y\}. \tag{2.2}$$

In other words, the set of chosen elements under budget $B(p, m)$ is exactly the set of $\succcurlyeq$-most preferred elements from $B(p, m)$. We say that $\succcurlyeq$ is a *rationalization for* h, and that $\succcurlyeq$ *rationalizes* h.

There are many subsidiary types of rationality: one can talk of *transitive-rationality* (for rationalization by a transitive relation $\succcurlyeq$) or *total-rationality* (for rationalization by a total relation $\succcurlyeq$), etc. A very important type of rationality is *regular-rationality*, in which there is a reflexive, transitive, and total rationalization $\succcurlyeq$. And then, of course, one can consider *utility-rationality*, in which there is a rationalization that is representable by a numerical utility function U on X,

$$h(p, m) = \{x \in B(p, m): \forall y_{y \in B(p, m)}\ U(x) \geqq U(y)\}, \tag{2.3}$$

for all $(p, m) \in \mathscr{B}$. Even more demanding, of course, would be to seek a rationalization that has a continuous, strictly concave, and strictly monotone utility representation. For brevity, we will call this *special-rationality*.

We say that a choice is *exhaustive* (on $\mathscr{B}$) if it satisfies the budget equality

$$p \cdot x = m \tag{2.4}$$

for all $x \in h(p, m)$ and all $(p, m) \in \mathscr{B}$. Of course, if only price vectors p and commodity vectors x are observed, but not incomes m, then for each such p and x we may define an $m = p \cdot x$, and then h is automatically exhaustive.

The main result of this paper can be viewed in three ways. First, it gives an empirical test for the existence of a continuous (and generically C^∞), strictly concave, and strictly monotone utility-rationalization for any finite set of demand data (Theorem 1). Second, it gives a procedure for constructing such a rationalization (Theorem 2). Third, it shows that certain low types of rationality are actually equivalent to the much higher special-rationality type (Theorem 3).

3. Regular and Special Rationality

First we state some definitions. Following [15, 16] we define the binary relation S on X by: for all $x, y \in X$,

$$xSy \quad \Leftrightarrow \quad \exists B_{B \in \mathscr{B}} x \in h(B)\ \&\ x \neq y \in B. \tag{3.1}$$

Let H be the transitive closure[1] of S. Then Houthakker's *Strong Axiom of Revealed Preference* can be stated as

$$H \text{ is asymmetric.} \tag{3.2}$$

It is known from Richter [14, 15] that, for demand *functions*, the Strong Axiom is equivalent to regular-rationality.

We say that a set $A \subseteq R^n$ is *generic* if it includes an open dense set whose complement is null (of Lebesgue measure zero). Equivalently, the complement of a generic set is small in the sense that it is a subset of a closed nowhere dense null set. We say that a property holds *generically*, if it holds on a generic set.

We will prove that a very high type of rationality follows from just the Strong Axiom.

THEOREM 1. *Let h be an exhaustive demand function defined on a finite subset $\mathscr{B}$ of $\mathscr{C}$. Then h has a special-rationalization U if and only if h satisfies the Strong Axiom of Revealed Preference. Furthermore, when such a U exists, it can be chosen to be defined on all of R^n, and be generically C^∞.*

Remark 1. There is no hope, however, of obtaining differentiability of rationalizations. In the two observations of Fig. 1a,[2] for example, we have $x = h(\bar{p}_1, \bar{p}_2, \bar{m}) = h(\hat{p}_1, \hat{p}_2, \hat{m})$. If there were a differentiable utility rationalization, then Lagrange's theorem on constrained maximization would guarantee the existence of $\bar{\lambda}$ and $\hat{\lambda}$ satisfying $D_i u(x_1, x_2) = \bar{\lambda}\bar{p}_i = \hat{\lambda}\hat{p}_i$ for $i = 1, 2$. Both $\bar{\lambda}$ and $\hat{\lambda}$ must be zero; otherwise $D_1 u(x_1, x_2)/D_2 u(x_1, x_2) = \bar{p}_1/\bar{p}_2 = \hat{p}_1/\hat{p}_2$, which contradicts Fig. 1a, since the budget lines have different slopes. So $D_i u(x_1, x_2) = 0$ for $i = 1, 2$. Since u is concave, x globally maximizes u, contradicting the strict monotonicity of u.

Of course, without monotonicity, the differentiable strictly concave utility $u(x_1, x_2) \equiv (x_1 - 1)^2 + (x_2 - 1)^2$ does rationalize Fig. 1a. But even

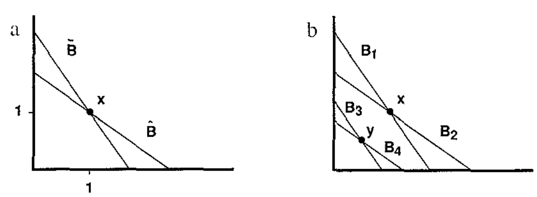

FIGURE 1

[1] The *transitive closure* of a relation R is defined as the smallest transitive relation including R

[2] The figure is essentially the same as in Chiappori and Rochet [3] The interpretation is different, since they use a weaker notion of rationality (cf Section 5 below)

without requiring monotonicity, reasoning similar to the above shows that no differentiable strictly concave utility can rationalize the four observations of Fig. 1b.

Nevertheless, despite Remark 1, we will show that a utility-rationalization can be found that is generically C^∞.

Remark 2. Theorem 1 gives an empirical test (the Strong Axiom) for determining whether a finite set of demand data can be rationalized by a continuous, strictly concave, and strictly monotone utility function. For it is clear that one can design algorithms to test, on any finite data set, whether the Strong Axiom holds. The next theorem also makes that clear, since it shows that satisfaction of the Strong Axiom is equivalent to solvability of a certain system of linear equalities and inequalities; and algorithms to test such solvability, and obtain solutions, are well known.[3] In addition, it should be noted that the proof of Theorem 2 below will provide a constructive method for obtaining special-rationalizations from the system's solutions.

A proof of the main part of Theorem 1, for compact domains X, was given in Matzkin [12]. Here we present a very different proof, based on the fact that Theorem 1 is an immediate corollary of Theorem 2, which is proved in Section 4.

THEOREM 2. *Let h be an exhaustive demand function on a finite subset $\mathscr{B} = \{(p^1, m^1), ..., (p^k, m^k)\}$ of $\mathscr{C}$. Let $x^i \in h(p^i, m^i)$ for $i = 1, ..., k$. Then the following statements are equivalent*:

(a) *h satisfies the Strong Axiom of Revealed Preference.*

(b) *There exists a continuous, strictly concave, and strictly monotone function U rationalizing h on $(X, \mathscr{B})$. (I.e., h is special-rational.) (Optionally, U can be chosen to be defined on all of R^n, and be generically C^∞.)*

(c) *There exist real numbers u^i, u^j and λ^i $(i, j = 1, ..., k)$ satisfying*:

$$u^i + \lambda^i p^i [x^j - x^i] > u^j \qquad \textit{for all} \quad i, j = 1, ..., k \textit{ with } x^i \neq x^j \qquad (3.3a)$$

$$\lambda^i > 0 \qquad \textit{for all} \quad i = 1, ..., k \qquad (3.3b)$$

$$u^i = u^j \qquad \textit{for all} \quad i, j = 1, ..., k \textit{ with } x^i = x^j. \qquad (3.3c)$$

(Optionally, the λ^i can be chosen so that $\lambda^i p^i \neq \lambda^j p^j$ for $i \neq j$.) (3.3d)

(d) *h is regular-rational.*

[3] Cf. the discussion of Fourier elimination in Stoer and Witzgall [21, Theorem 1.1.9 and Sects 1 2 and 1 3]

Remark 3. Conditions (c) are a strengthening[4] of Afriat's inequalities [1, p. 73 (Theorem)]. To prove that (c) implies (b) we follow his method, but modify his proof to obtain a stronger result. The modifications are necessary because our definition of rationality is much stricter than this, and because we insist on strict concavity of the rationalizing utility. See Section 5 below for a more detailed comparison between his work and ours.

4. Proof of Theorem 2

Proof of Theorem 2. That (a) implies (c) is the assertion of Lemma 1 below. That (c) implies (b) is the assertion of Lemma 2 below. A fortiori, (b) (even without the parenthetical options) implies (d). That (d) implies (a) is known from Richter [14]. ∎

The statement and proof of Lemma 1 are complicated by the fact that we can have data with $x^i = x^j$ even when the budget sets corresponding to i and j are different. Indeed, the reader should note that the statement and proof are much simpler in the special case where the demand function is invertible, so that (4.1c), (4.2c), and (4.3b) disappear from the proof below.

In Lemma 1 we prove that u^i, λ^i exist as in (3.3): we show that otherwise the Theorem of the Alternative would give choice cycles contradicting the Strong Axiom.

Lemma 1. *Under the hypotheses of Theorem* 2, *if h satisfies the Strong Axiom of Revealed Preference, then there exist u^i, u^j, λ^i satisfying* (3.3).

Proof. We seek real numbers u^i, u^j, λ^i $(i, j = 1, ..., k)$ that solve

$$u^i - u^j - \lambda^i p^i[x^i - x^j] > 0 \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i \neq x^j \qquad (4.1a)$$

$$\lambda^i > 0 \quad \text{for all} \quad i = 1, ..., k \qquad (4.1b)$$

$$u^i - u^j = 0 \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i = x^j, \qquad (4.1c)$$

with $\lambda^i p^i \neq \lambda^j p^j$ for $i \neq j$. Defining $\alpha^{ij} \equiv p^i[x^i - x^j]$, we can rewrite this as

$$u^i - u^j - \lambda^i \alpha^{ij} > 0 \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i \neq x^j \qquad (4.2a)$$

$$\lambda^i > 0 \quad \text{for all} \quad i = 1, ..., k \qquad (4.2b)$$

$$u^i - u^j = 0 \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i = x^j, \qquad (4.2c)$$

with $\lambda^i p^i \neq \lambda^j pj$ for $i \neq j$. Let K be the number of pairs (i, j) with $x^i \neq x^j$.

[4] Afriat used a weak inequality in his analogue of (3.3a).

Then we can rewrite this as

$$Ar > 0 \tag{4.3a}$$

$$Cr = 0, \tag{4.3b}$$

where $r \equiv (u^1, ..., u^k, \lambda^1, ..., \lambda^k)$, and where A and C are matrices with $2k$ columns, defined as follows. (An example of the matrices A and C follows in (4.4). It may be helpful to refer to that example while reading the defintions of A and C.)

Matrix A has $K+k$ rows. The first K rows correspond to the left hand side of (4.2a), with 0 in all positions, except for a 1 in position i, and a -1 in position j, and $-\alpha^{ij}$ in position $k+i$; and the last k rows correspond to the left hand side of (4.2b), with 0 in all positions, except for a 1 in position $k+i$.

Matrix C corresponds to the left hand side of (4.2c). (If C is not empty, there are obvious redundancies we can eliminate without changing the solutions of (4.3). For example, if $u^i - u^j = 0$ is one of the lines of (4.2c), then we can eliminate the row corresponding to $u^j - u^i$.) Let L be the number of rows in C.

As an example, the matrices A and C for just four observations, with x^1, x^2, and x^3 distinct, and $x^3 = x^4$, look like this:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & -\alpha^{12} & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -\alpha^{13} & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -\alpha^{14} & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -\alpha^{21} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -\alpha^{23} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -\alpha^{24} & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & -\alpha^{31} & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & -\alpha^{32} & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & -\alpha^{41} \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -\alpha^{42} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.4a}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.4b}$$

To prove that a solution vector r exists for (4.3), suppose not; we obtain a contradiction as follows. Since no such r exists, then, by a Theorem of the

Alternative[5] there exist a $K+k$-dimensional vector v and an L-dimensional vector z such that[6]

$$v'A + z'C = (0, ..., 0) \tag{4.5a}$$

$$v \geqslant 0. \tag{4.5b}$$

If some row i of C has a corresponding $z_i < 0$, then we can, without changing the solutions of (4.3), replace that row by its negative, and replace z_i in (4.5a) by $-z_i$. So in (4.5) we can without loss of generality assume that

$$v \geqslant 0 \qquad \text{and} \qquad z \geqq 0. \tag{4.5c}$$

We will refer to the rows of A according to their α-terms: thus $\text{Row}^A(i, j)$ is the row containing the term $-\alpha^{ij}$. If a $\text{Row}^A(i, j)$ has a positive v-multiplier in (4.5), then we call $\text{Row}^A(i, j)$ a *weighted* row. We will refer to the ith column of matrix A or C as $\text{Col}^A(i)$ or $\text{Col}^C(i)$.

Define γ_i terms by

$$z'C = (\gamma_1, ..., \gamma_k, 0, ..., 0). \tag{4.6}$$

Since $v \geqslant 0$, we can without loss of generality suppose $v_1 > 0$. Then (4.5) implies that we cannot have $-\alpha^{1i} > 0$ for all $i = 1, ..., K$ (since $v' \text{Col}^A(k+1) = 0$). So without loss of generality we can assume that $-\alpha^{12} \leqq 0$, hence $x^1 S x^2$. We will now show that there exists a weighted row $i \neq 1$ with $-\alpha^{2i} \leqq 0$, hence

$$x^2 S x^i. \tag{4.7}$$

The second component, -1, of the first row of A clearly guarantees (by (4.5)) that either (i) there is some weighted row of A with 1 as its second component, or (ii) $\gamma_2 > 0$. In case (i), say $\text{Row}^A(2, i)$ is weighted and its second component is 1. Then (4.5) clearly implies that we cannot have $-\alpha^{2j} > 0$ for all j, since $v' \text{Col}^A(k+2) = 0$. So there is some j with $-\alpha^{2j} \leqq 0$, hence $x^2 S x^j$, so (4.7) holds in case (i). In case (ii), we have $\gamma_2 > 0$. Then we will prove below

$$\text{There exists some } j = 1, ..., k \text{ such that } \gamma_j < 0 \text{ and } x^j = x^2. \tag{4.8}$$

[5] Since $r = 0$ satisfies (4.3b), we can apply Rockafellar [18, p. 198 (Theorem 22.2)-199], after writing the equality (4.3b) as the pair of inequalities $Cr \leqq 0$ and $Cr \geqq 0$. (It can easily be verified that the effect of this is to remove Rockafellar's nonnegativity requirements on the multipliers corresponding to (4.3b).)

[6] By $v \geqslant 0$ we mean $v \geqq 0$ and $v \neq 0$

It follows then from (4.5) that there is some weighted $\mathrm{Row}^A(j, i)$ of A with 1 in $\mathrm{Col}^A(j)$. Again by (4.5), it cannot be that all such rows $\mathrm{Row}(j, i)$ have $-\alpha^{ji} > 0$. So we can choose an i such that $-\alpha^{ji} \leqq 0$, hence $x^j S x^i$. Since $x^j = x^2$ by (4.8), this proves (4.7).

Continuing in this fashion by finite induction, we obtain $x^1 S x^2 S x^3 S \cdots$. Since there are only finitely many columns in the matrix A, this forces a contradiction of the Strong Axiom, which prevents any "cycling" back to previous x^i. And this contradiction completes the proof of Lemma 1, subject to verification of (4.8).

To prove (4.8), let J be the set of j with $x^j = x^2$. Then if (4.8) were not true, we would have

$$z' \,\mathrm{Col}^C(j) \geqq 0 \qquad \text{for every} \quad j \in J. \tag{4.9}$$

By our hypothesis (ii),

$$z' \,\mathrm{Col}^C(2) = \gamma_2 > 0. \tag{4.10}$$

So (4.9) and (4.10) imply

$$0 < \sum_{j \in J} \gamma_j \tag{4.11a}$$

$$= \sum_{j \in J} z' \,\mathrm{Col}^C(j) \tag{4.11b}$$

$$= \sum_{j \in J} \sum_{i=1}^{L} z_i (\mathrm{Col}^C(j))_i \tag{4.11c}$$

$$= \sum_{i=1}^{L} z_i \sum_{j \in J} (\mathrm{Col}^C(j))_i \tag{4.11d}$$

$$= 0. \tag{4.11e}$$

The justification for (4.11e) is that, for each i, $\sum_{j \in J} (\mathrm{Col}^C(j))_i = 0$, because in each row i of C there are only two columns $j \in J$ with nonzero entries: one 1 and one -1. The contradiction (4.11) proves (4.8).

Finally, because the inequalities (4.2a, b) are strict, the λ's can clearly be chosen so that $\lambda^i p^i \neq \lambda^j p^j$ for $i \neq j$. ∎

Remark 4. Instead of basing our proof of Lemma 1 on a Theorem of the Alternative, a proof could be obtained through an algorithm very much in the spirit of Varian's algorithm [23].

In Lemma 2 we use the u^i, λ^i from Lemma 1 to obtain a strictly concave, strictly monotone, rationalizing utility on all of R^n.

LEMMA 2. *Under the hypotheses of Theorem 2, if there exist u^i, u^j, λ^i satisfying* (3.3), *including the optional part, then h has a special rationalization U, defined on all of R^n.*

Proof. Part A. Definition of U. Let u^i and λ^i ($i=1, ..., k$) satisfy the inequalities (3.3). Since there are only finitely many inequalities, clearly there exists an $\varepsilon_0 > 0$ such that

$$u^i + \lambda^i p^i[x^j - x^i] - \varepsilon_0 > u^j \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i \neq x^j \tag{4.12a}$$

$$\lambda^i > 0 \quad \text{for all} \quad i = 1, ..., k \tag{4.12b}$$

$$u^i = u^j \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i = x^j. \tag{4.12c}$$

Now let $T > 0$, and define $g\colon R^n \to R^1$ by[7]

$$g(x_1, ..., x_n) = (x_1^2 + \cdots + x_n^2 + T)^{1/2} - T^{1/2}. \tag{4.13}$$

Then

$$g(x) > 0 \quad \Leftrightarrow \quad x \neq 0 \tag{4.14a}$$

$$g(x) = 0 \quad \Leftrightarrow \quad x = 0 \tag{4.14b}$$

$$\left[\frac{\partial g}{\partial x_i}(x)\right] < 1 \quad \text{for all} \quad x \text{ and } i = 1, ..., n \tag{4.14c}$$

$$g \text{ is strictly convex} \tag{4.14d}$$

$$g \text{ is differentiable.} \tag{4.14e}$$

Then by (4.12) we can pick $\varepsilon > 0$ so small that

$$u^i + \lambda^i p^i[x^j - x^i] - \varepsilon g(x^j - x^i) > u^j \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i \neq x^j \tag{4.15a}$$

$$\lambda^i > 0 \quad \text{for all} \quad i = 1, ..., k \tag{4.15b}$$

$$u^i = u^j \quad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i = x^j. \tag{4.15c}$$

Now, for each $i = 1, ..., k$, define $\phi_i\colon R^n \to R^1$ by

$$\phi_i(x) \equiv u^i + \lambda^i p^i[x - x^i] - \varepsilon g(x - x^i). \tag{4.16}$$

Since g is strictly convex, each ϕ_i is strictly concave. And clearly

$$\phi_i(x^i) = u^i \qquad (i = 1, ..., k). \tag{4.17}$$

[7] Here the superscripts are exponents, not indices.

Now define $U: R^n \to R^1$ by

$$U(x) \equiv \min\{\phi_i(x): i=1, ..., k\} \tag{4.18}$$

for all $x \in X$. As the minimum of finitely many strictly concave functions, U is strictly concave.

Part B. Monotonicity of U. We will choose ε to guarantee strict monotonicity. Since (4.18) defines U as the minimum of finitely many ϕ_i functions, it clearly suffices to show that each ϕ_i has everywhere a strictly positive partial derivative. From (4.16), the partial derivatives are given by

$$\begin{aligned} D_j\phi_i(x) &= \lambda^i p^i_j - \varepsilon D_j g(x - x^i) \qquad \text{for all} \quad j=1, ..., n && (4.19\text{a}) \\ &> \lambda^i p^i_j - \varepsilon 1 \qquad \text{(by (4.14c))}. && (4.19\text{b}) \end{aligned}$$

Since there are only finitely many indexes $i = 1, ..., k$, we can pick ε so small that this is positive for all $i = 1, ..., k$ and all $j = 1, ..., n$.

Part C. U rationalizes. As a first step in proving that U rationalizes h, we will show that

$$U(x^j) \geqq u^j \qquad \text{for all} \quad j = 1, ..., k. \tag{4.20}$$

If (4.20) were not true, then we would have

$$\begin{aligned} u^j &> U(x^j) && (4.21\text{a}) \\ &= \phi_i(x^j) \qquad \text{for some} \quad i=1, ..., k \qquad \text{(by (4.18))} && (4.21\text{b}) \\ &= u^i + \lambda^i p^i[x^j - x^i] - \varepsilon g(x^j - x^i) \qquad \text{(by (4.16))} && (4.21\text{c}) \end{aligned}$$

which contradicts inequality (4.15a) if $x^i \neq x^j$, and contradicts (4.15c) if $x^i = x^j$. So (4.20) holds.

(Although it is not needed in our proof, we note here that, in fact, equality holds in (4.20). For

$$\begin{aligned} \exists i_{i=1, ..., k}\, U(x^j) &= \phi_i(x^j) \qquad \text{(by (4.18))} && (4.22\text{a}) \\ &\leqq \phi_j(x^j) \qquad \text{(by (4.18))} && (4.22\text{b}) \\ &= u^j \qquad \text{(by (4.17))}. && (4.22\text{c}) \end{aligned}$$

The equality then follows from (4.20) and (4.22).)

Next we show that U utility-rationalizes h. It is clearly sufficient to show that, for each $i = 1, ..., k$,

$$\forall y_{p^i y \leqq m^i}\, y \neq x^i \Rightarrow U(x^i) > U(y). \tag{4.23}$$

Now for such y,

$$\begin{aligned} U(y) &= \min\{\phi_j(y): j=1, ..., k\} \qquad \text{(by (4.18))} && (4.24a) \\ &= \min\{u^j + \lambda^j p^j[y-x^j] - \varepsilon g(y-x^j): j=1, ..., k\} \\ &\qquad \text{(by (4.16))} && (4.24b) \\ &\leqq u^i + \lambda^i p^i[y-x^i] - \varepsilon g(y-x^i) && (4.24c) \\ &< u^i \qquad \text{since } y \neq x^i \qquad \text{(by (4.15b), the budget equality,} \\ &\qquad \text{and (4.14a))} && (4.24d) \\ &\leqq U(x^i) \qquad \text{(by (4.20)).} && (4.24e) \end{aligned}$$

So (4.23) holds, and U utility-rationalizes h.

Part D. Genericity of infinite differentiability. Define

$$E \equiv \{x \in R^n: U \text{ is not } C^\infty \text{ at } x\}. \tag{4.25}$$

Clearly

$$E \subseteq \bigcup \{E_{ij}: i, j=1, ..., k \text{ and } i \neq j\}, \tag{4.26a}$$

where

$$E_{ij} \equiv \{x \in R^n: \phi_i(x) = \phi_j(x)\}. \tag{4.26b}$$

Define

$$f_{ij} \equiv \phi_i - \phi_j, \tag{4.27}$$

so

$$E_{ij} = f_{ij}^{-1}(0). \tag{4.28}$$

Now it is easily checked that for all small enough $\varepsilon > 0$ (cf. (4.15, 16)) we have, for all $x \in R^n$ and $i \neq j$,

$$Df_{ij}(x) \neq (0, ..., 0). \tag{4.29}$$

So it follows from the Implicit Function Theorem[8] that

$$f_{ij}^{-1}(0) \text{ is an } (n-1)\text{-dimensional } C^\infty \text{ submanifold of } R^n, \tag{4.30}$$

hence E_{ij} is the complement of an open dense subset of R^n. Then E, as a subset of a finite union of such sets, is the complement of a generic set. ∎

[8] Cf Guillemin and Pollack [5, p 21, Preimage Theorem]; Kahn [9, p 69, Proposition 3 1].

5. Comparison with Other Results

To put our results in context, we first mention two other notions of rationality. If we replace the "$=$" in (2.3) by "$\subseteq$" or "$\supseteq$" we obtain two defintions of *semirationality* which we call *subsemirationality* and *suprasemirationality*, respectively. They are clearly much weaker concepts than the rationality notion of (2.3). Note, for example, that every constant function is a subsemirationalization of *any* demand correspondence.

We can use these definitions to clarify two main lines of research. Although the terminology may have been different, the revealed preference work of Samuelson [19, 20], Houthakker [6], Uzawa [22], and Richter [14] has worked primarily with the stricter notion of rationality. On the other hand, a line of work by Afriat [1, 4], Diewert [4], and Varian [23] has used the weaker notion of subsemirationality.

Afriat [1] stated several conditions on finite sets of demand data from which he proved subsemirationality ("utility consistency" or "utility hypothesis" in his terminology). Afriat showed that his consistency conditions were also equivalent to "normal utility consistency," which in our terminology would mean concave-utility-subsemirationality. Since only weak concavity was required, any constant function would again be such a subsemirationalization.[9]

Afriat was interested in not just proving existence of a utility subsemirationalization, but also in providing a method for actually calculating such a function. The particular method he used was further developed by Diewert, who obtained such a function by solving a linear programming problem. Varian restated Afriat's result and construction in terms of a Generalized Axiom of Revealed Preference, which is weaker than the Strong Axiom; and he gave an algorithm to find a solution for the u^i, λ^i that, when the Generalized Axiom holds, satisfies Afriat's inequalities.

Our conclusions sharpen these subsemirationality results. First, we provide rationality in the full sense of (2.3).[10] Second, we guarantee strict concavity and strict monotonicity of some utility-rationalization. (The earlier constructions were never strictly concave.) We use this to show that special rationality is equivalent to regular rationality. As in the earlier results, our conditions are algorithmically testable, and our proof also shows how to actually construct a utility-rationalization.

Recently Chiappori and Rochet [3] have strengthened the Strong Axiom hypothesis by adding what amounts to invertibility of the observed

[9] And would be a counterexample to the claim (Afriat [1, pp 69, 74 (Corollary)]) that cyclical consistency is a necessary consequence of utility consistency

[10] Of course, even a subsemirationalization by a *strictly* concave function would be a full rationalization, since we are assuming that all budgets are convex

demand function. Then they showed that one can obtain, on any compact subset of $R^n_{\geqq}$, a C^∞, monotone, subsemi-utility-rationalization. Although their formal definition of rationality is only subsemi-rationality, their strict concavity conclusion actually yields what we have called rationality.

After their results were published, we realized that the methods of our Theorem 2 could be used to strengthen their conclusion: As we show in Theorems 1^∞ and 2^∞, under their special invertibility hypothesis, one can obtain a C^∞ utility-rationalization on all of R^n.

We call a demand function $h: \mathscr{B} \to X$ (*homogeneously*) *invertible* when, for all (p, m), $(p', m') \in \mathscr{B}$: if $h(p, m) = h(p', m')$, then (p, m) is a positive scalar multiple of (p', m')—denoted $(p, m) \propto (p', m')$.

THEOREM 1^∞. *Let h be an exhaustive demand function defined on a finite subset $\mathscr{B}$ of $\mathscr{C}$. Then h has a C^∞ special rationalization defined on all of R^n if and only if h satisfies the Strong Axiom of Revealed Preference and is invertible.*

Theorem 1^∞ is an immediate corollary of

THEOREM 2^∞. *Let h be an exhaustive demand function on a finite subset $\mathscr{B} = \{(p^1, m^1), ..., (p^k, m^k)\}$ of $\mathscr{C}$. Let $x^i \in h(p^i, m^i)$ for $(i = 1, ..., k)$. Then the following statements are equivalent*:

(a) *h satisfies the Strong Axiom of Revealed Preference and is invertible.*

(b) *There exists a C^∞, strictly concave, and strictly monotone function U rationalizing h on $\mathscr{B}$. (I.e., h is special-rational.) And U can be defined on all of R^n.*

(c) *There exist real numbers u^i, u^j and λ^i $(i, j = 1, ..., k)$ satisfying*:

$$u^i + \lambda^i p^i [x^j - x^i] > u^j \qquad \text{for all} \quad i, j = 1, ..., k \text{ with } x^i \neq x^j \qquad (5.1a)$$

$$\lambda^i > 0 \qquad \text{for all} \quad i = 1, ..., k \qquad (5.1b)$$

$$x^i = x^j \Rightarrow (p^i, m^i) \propto (p^j, m^j) \qquad \text{for all} \quad i, j = 1, ..., k \qquad (5.1c)$$

(d) *h is regular-rational and invertible.*

Proof. To prove that (a) implies (c), note that (5.1c) follows from invertibility; the rest follows as in Lemma 1, where the proof is now simplified by assuming that the matrix C no longer appears (cf. (4.3b)) (this simplification is justified because, by invertibility of h, we can assume that all budgets (p^i, m^i) are distinct). To prove that (c) implies (b), we first obtain the function U as in the proof of Lemma 2 (4.18). Then, because (5.1c) guarantees invertibility of h, we can apply to our U the same convolution methods that [3] applied to their W, and we obtain a C^∞ function V that is strictly concave and monotone (because our U had those properties) and is such that, for every i, $DV(x^i) = \lambda^i p^i$. Since V is strictly concave, it is easy to check that it rationalizes the observations. ∎

6. Applications

Our results has many applications. We mention a few.

(i) As already noted, Theorems 1 and 2 give necessary and sufficient conditions in empirically testable forms, for the existence of special utility-rationalizations on finite data sets. And Theorem 2 shows one way to construct such utilities.

(ii) Although Theorem 1 concerns just finitely many observations, it can be used to deduce rationality properties for arbitrary sets of demand data. This is demonstrated in Richter's new rationality results for noncontinuous and continuous demands [17].

(iii) Our results can yield a very simple proof of the classical Slutsky conditions [8, Theorem 1]. This is shown in [17] by employing our Theorem 1 to deduce rationality properties for arbitrary demand data, as explained in (ii), and using those properties together with a general convexity lemma.

(iv) Given a finite number of budget-demand observations, can we interpolate between them a continuous demand function that comes from a utility-rational consumer? Theorem 1 shows that this is possible if and only if the observations satisfy the Strong Axiom. In particular, the Strong Axiom is well known to be necessary for regular-rationality. And if the Strong Axiom holds, then Theorem 1 shows that the demand can be utility-rationalized by a continuous (strictly monotone, strictly concave) function, which by standard results must generate a continuous demand function. (This is a continuous analogue of the C^∞ version for compact sets, given in the Corollary of Chiappori and Rochet [3]).

(v) Theorem 1 also allows us to sharpen Mas-Colell's approximation result for continuous demand functions. He showed that income Lipschitzian demands satisfying a certain boundary condition can be rationalized by a unique continuous preference; and furthermore, this preference can be approximated by monotone, concave, subsemirationalizations on finite subsets of demand data (Mas-Colell [11], remarks preceding Theorem 4). Now we can replace his application of Afriat's result by an application of our Theorem 1; this strengthens his result so that the approximating preferences are full rationalizations, and they have utility functions that are strictly concave and strictly monotone.

(vi) If we combine Theorem 1 with Kim's recent result [10], that the Strong Axiom is equivalent to semitransitive-rationality (and to pseudotransitive-rationality), then we immediately obtain

THEOREM 3. *Let h be an exhaustive demand function defined on a finite subset $\mathscr{B}$ of $\mathscr{C}$. Then h is special-rational if and only if h is semitransitive-rational (equivalently, pseudotransitive-rational).*

For finite sets of data, then, empirical tests cannot distinguish between certain weak types of nontransitive rationality and the much stronger special-rationality.

(vii) Theorem 2 has been used by Brown and Matzkin [2] to characterize any finite set of endowment-price points lying on the graph of the equilibrium correspondence of any economy in which consumers' preferences can be represented by strictly concave and strictly monotone utility functions.

(viii) Theorem 1 and the technique employed in our proof of Lemma 2 to construct a rationalization have been employed by Matzkin [13] to characterize choice data generated by maximization of strictly concave and strictly monotone functions, subject to a variety of nonlinear choice sets.

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Invited Review

Revealed preference theory: An algorithmic outlook

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ABSTRACT

Revealed preference theory is a domain within economics that studies rationalizability of behavior by (certain types of) utility functions. Given observed behavior in the form of choice data, testing whether certain conditions are satisfied gives rise to a variety of computational problems that can be analyzed using operations research techniques. In this survey, we provide an overview of these problems, their theoretical complexity, and available algorithms for tackling them. We focus on consumer choice settings, in particular individual choice, collective choice and stochastic choice settings.

1. Introduction

1.1. Motivation

Our world is full of choices. Before we step outside the door in the morning, we have already chosen what to eat for breakfast and which clothes to wear. For the morning commute, we decide how to travel, by what route, and whether we will pick up coffee along the way. Dozens of small choices are made before it is even time for lunch, and then there are the less frequent, but more important decisions like buying a car, moving to a new home, or setting up retirement savings. Neoclassical economists hypothesize that such consumption choices are made so as to maximize utility. Given this hypothesis, it follows that each choice tells us something about the decision maker. In other words, choices *reveal preferences*, and thereby provide information about an underlying utility function. As we observe the choices of a decision maker over time, we can piece together more and more information. Given this information about choices made, a number of questions naturally arise:

i) Does there exist a utility function which is consistent with the observed choices?
ii) When a consistent utility function exists, does there exist one in a prespecified class?
iii) When no consistent utility function exists, how close are the observed choices to being consistent?

These questions belong to the domain of *revealed preference theory*, pioneered by Samuelson (1938, 1948). In this theory, it is usual to formulate a minimum set of prior assumptions, also known as *axioms*, which are based on a theory of choice behavior. Thus, revealed preference characterizations are defined as conditions on the observed choices of decision makers. This approach allows for direct tests of the decision models, without running the risk that excessively strong functional (mis)specifications lead to rejections of the model.

Testing the axioms of revealed preference theory is a topic at the interface of economics and operations research. We focus on the algorithmic aspects of solving the corresponding optimization/decision problems, and we highlight some of the issues of interest from the operations research viewpoint. In particular, we examine algorithms that can be used to test whether observed consumer choices satisfy certain revealed preference conditions. We also look at the tractability, that is, the computational complexity of algorithms for answering these questions. Following the classical framework of computational complexity (see, for instance, Garey & Johnson, 1979 or Cormen, Leiserson, Rivest, & Stein, 2001), we focus on worst-case time-bounds of algorithms. We are especially interested in whether a particular question is easy (that is, solvable in polynomial time) or difficult (NP-HARD), and what the best-known method is for answering the question.

Let us first motivate this computational point of view. In a very general way, it is clear that computational issues have become increasingly important in all aspects of science, and economics is no exception. This is reflected, in particular, in the economic literature on revealed preference, where computational challenges are frequently and explicitly mentioned. We illustrate this claim with three quotes from recent papers.

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Echenique, Lee, and Shum (2011) write:

> "Given [that calculating money pump costs can be a huge computational task], we check only for violations of GARP that involve cycles of limited length: lengths 2, 3, and 4."

Choi, Kariv, Müller, and Silverman (2014) write (in the online appendix):

> "Since the algorithm is computationally very intensive, for a small number of subjects we report upper bounds on the consistent set."

Kitamura and Stoye (2014) write:

> "It is computationally prohibitive to test stochastic rationality on 25 periods at once. We work with all possible sets of eight consecutive periods, a problem size that can be very comfortably computed."

These quotes signify the need for fast algorithms that can test rationality of choices made by an individual (or a group of individuals), or at least to better understand the tractability of these underlying questions.

Another trend that emphasizes the relevance of efficient computations in the domain of revealed preference is the ever-increasing size of datasets. As in many other fields of social and exact sciences, and as underlined by the pervasiveness of buzzwords such as "big data" and "data science", more and more information is available about actual choices of decision makers. As a striking example, it is now commonplace for brands or large retailers to track the purchases of individual consumers or households. This activity yields numerous datasets with sizes far beyond those provided by laboratory experiments. This only reinforces the need for efficient methods, in order to be able to tackle and to draw meaningful conclusions from huge datasets. For example, Cherchye et al. (2017a) use revealed preference models to study food choices. The sample they analyze contains records of all grocery purchases of 3645 individuals over a period of 24 months. It is extracted from the Kantar Worldpanel, which records the purchases of 25,000 households. Long-running longitudinal studies actually provide large datasets of household consumption and other economic indicators. Cherchye, Demuynck, De Rock, and Vermeulen (2017b) identify intrahousehold decision structures using such large datasets.

In view of these considerations, there is a quickly growing body of work on computation and economics. As mentioned above, our objective is to give an overview of algorithmic problems arising in revealed preference theory. Due to the wide range of choice situations to which revealed preference has been applied, providing a comprehensive overview is not a realistic goal. In this paper, we focus on algorithmic results concerning tests of rational behavior in consumer choice settings. For different discussions of the topic, we refer the reader to the recent monograph on the theory of revealed preference by Chambers and Echenique (2016), and to a survey by Crawford and De Rock (2014) on empirical revealed preference; an earlier overview can be found in Houtman (1995). Finally, we should note that certain aspects of revealed preference theory, as a way of explaining choice behavior, have also been criticized; see, e.g., the works of Hausman (2000) and Wong (2006).

1.2. Preference modeling and utility theory

Before we close this introductory section, we find it useful to formulate a few comments on the relations between the stream of literature that we cover in this paper, and the literature on preference modeling and utility-based decision making, as they have classically been handled in operations research (OR) and, more recently, in artificial intelligence (AI). Our goal is obviously not to survey these huge and active fields of research. Rather, we simply intend to clarify some of the similarities and differences that exist between the "economic" setting of revealed preference theory, and an "operations research" or "artificial intelligence" perspective which may be more familiar to readers of this journal.

Many of the results surveyed in this paper express conditions for the existence of a utility function which represents the preferences revealed through the choices made by consumers. Most of these results have been published by economists. On the other hand, in operations research and in decision theory, there is a long tradition of building utility functions (sometimes called "value functions" in the deterministic setting) based on information provided by one or several decision makers. Classical references are, for instance, Fishburn (1970), Keeney and Raiffa (1976). Typically, in such settings, the preferences of the decision maker are expressed by a limited number of pairwise comparisons of alternatives, or by rankings of the alternatives on several criteria. The objective is then to build a utility function which is coherent with the expressed preferences, and which can be used, for instance, in order to evaluate each and every alternative on a numerical or ordinal scale, or to evaluate alternatives that have not yet been seen. The utility functions under consideration may be as simple as a (weighted) sum of criteria, or may be selected within a parameterized class of functions whose parameters are to be determined. This type of approach has been extensively investigated, in particular, by researchers interested in multiple criteria problems with discrete alternatives (MCDA) (see, e.g., Greco, Ehrgott, & Figueira, 2016, and in particular Bouyssou & Pirlot, 2016; Dyer, 2016; Moretti, Öztürk, & Tsoukiàs, 2016; Siskos, Grigoroudis, & Matsatsinis, 2016 for recent surveys of closely related topics; see also Corrente, Greco, Matarazzo, & Słowiński, 2016 for extensions), or in conjoint analysis (see, e.g., Giesen, Mueller, Taneva, & Zolliker, 2010; Gustafsson, Herrmann, & Huber, 2007; Rao, 2014). More recently, similar questions have also been investigated in preference learning, a subfield of artificial intelligence (see, e.g., Corrente, Greco, Kadziński, & Słowiński, 2013; Fürnkranz & Hüllermeier, 2010).

Not surprisingly, all of these fields share a common theoretical basis, as well as many methodological concepts: preference relations, transitivity, pairwise comparisons, to name but a few. Nevertheless, they also all have their own specific purposes, assumptions, and applications, which lead to a variety of research questions and results. The objective of this survey is not to carry out a systematic comparison of these various settings. However, in order to avoid any confusion in the mind of the reader, we find it useful to briefly outline some of the most striking differences between revealed preference theory and other utility-based frameworks.

- The approaches proposed in OR and in AI are mostly *prescriptive* or *operational* in nature. Their main objective is to help an individual, or a group of individuals, to express and to structure their preferences, *so as to allow them to make informed decisions*. This is the case in MCDA, in conjoint analysis, and in preference learning. In contrast, the revealed preference literature is mostly *normative* (to the extent that it posits axioms of rational choice behavior) and *descriptive* (to the extent that it attempts *to test whether actual consumer choices are consistent* with the stated axioms), but *it is not meant to support any decision making process*. This is definitely a major distinguishing feature of revealed preference theory.
- As a corollary of the previous item, an objective frequently pursued in OR and in AI is *to explicitly build* ("assess", "elicit") a utility function which is compatible with the data; this is the case in multiattribute utility theory or in conjoint analysis, most noticeably. (Of course, some classical approaches to multicriteria decision making do not explicitly attempt to

build the utility function of the decision maker; this is the case, for instance, of the interactive methods developed by Zionts and Wallenius (1976, 1983), and of outranking methods such as described by Roy (1991).) On the other hand, in the economic literature, a main objective is *to check the coherence* of consumer choices with rationality axioms proposed in the theory. Hence, *building* a compatible utility function (sometimes called the "recovery" issue in economics) is usually not viewed as the primary outcome of the process. It should be noted, however, that the existence proofs provided for instance by Afriat (1967b) or Varian (1982) (see Section 3 hereunder) are constructive and provide an analytical expression of the utility function, when it exists. Predicting, or bounding the demand bundles associated with future prices is also a topic in interest in economics; see, e.g., (Blundell, 2005; Varian, 1982).

- In utility theory and in MCDA, the alternatives are often considered as "abstract", "unspecified" entities: most papers in this stream start with the assumption that the decision maker is facing "a set A of alternatives", or potential actions, but *the nature of these alternatives is not directly relevant* for the development of the theoretical framework (although, of course, the alternatives must be fully determined in any specific application of the theory); see (Dyer, 2016; Fishburn, 1970; Keeney & Raiffa, 1976). In conjoint analysis or in preference learning, the alternatives are represented as multidimensional vectors associated with product attributes or other measurable features. In revealed preference theory, on the other hand, the observations consist of *bundles of goods* and their *associated prices*: this assumption is crucial for the definition of the preference relation, as we explain next.
- In OR or AI, preferences among alternatives can be formulated in a variety of ways (e.g., through pairwise comparisons of alternatives), but are solely based on declarations of the decision maker. In revealed preference settings, on the contrary, the preferences between bundles are explicitly derived by the analyst from pairwise comparisons of the *prices of the bundles* purchased by the decision maker. As a consequence, goods and their prices play a central role and provide another distinguishing feature of the theory. In particular, many of the theorems regarding the existence of utility functions can be stated in terms of prices and quantities of goods.
- In MCDA, in conjoint analysis, or in preference learning, the procedure used to elicit the utility function often rests on the formulation of questions that can be submitted to the decision maker, possibly in an interactive, dynamic process; so, the *design of the most appropriate experiments* is an important issue to be tackled by the analyst, as it influences the relevance of the collected data and the efficiency of the elicitation process (see, e.g., Gustafsson et al., 2007; Rao, 2014; Riabacke, Danielson, & Ekenberg, 2012 for a discussion of such design issues). In revealed preference settings, the analyst usually faces the results of *uncontrolled experiments*, in the form of a database of observations which have been typically collected for other purposes (although the issue of experimental design is also discussed, for instance, in Blundell, 2005).
- As a consequence of the previous point, the datasets considered in the OR literature on preference modeling are often quite small, and computational complexity, or even algorithmic considerations have not been a main focus of attention in this area. (This is true, at least, for multiple criteria problems with discrete alternatives, as opposed to multiple criteria optimization problems which may feature an infinite set of feasible alternatives, such as a polyhedron described by linear inequalities, and which call for more efficient algorithmic approaches; see, e.g., (Wallenius et al., 2008) for a discussion of the growing importance of algorithmic issues in multicriteria decision making.) On the other hand, the databases to be handled in revealed preference studies are potentially huge, so that complexity issues naturally arise and have been considered, more or less explicitly, by various researchers. They provide the main theme to be covered in this paper.

As previously mentioned, in spite of the inherent differences outlined above, and in spite of the fact that the streams of research on utility-based decision making and on revealed preference have evolved in almost total separation, there remain some obvious commonalities between these topics. The objective of our survey, however, is not to establish a comparative study, but rather to provide the reader with an overview of fundamental results and of recent developments in the field of algorithmic revealed preference theory. We hope that this may lay the ground for future cross-fertilization between operations research and revealed preference theory.

1.3. Outline of the survey

We begin this survey by introducing key concepts in revealed preference theory, such as utility functions and preference relations, in Section 2. Next, in Section 3, we state the fundamental theorems that characterize rationalizability in revealed preference theory. We explicitly connect rationalizability with properties of certain graphs, and we state the worst-case complexity of algorithms that establish whether a given dataset satisfies a particular "axiom" of revealed preference. In Section 4, we look at various kinds of utility functions that have been considered in the literature, and we provide corresponding rationalizability theorems. Section 5 deals with goodness-of-fit and power measures, which respectively quantify the severity of violations and give a measure of how stringent the tests are. In Section 6, we explore collective settings, where the observed choices are the result of joint decisions by several individuals. Finally, in Section 7, we look at stochastic preference settings where the decision maker still attempts to maximize her utility, but her preferences are not necessarily constant over time. Instead, the decision maker has a number of different utility functions, and the function that she maximizes at any given time is probabilistically determined. We conclude in Section 8.

2. Preliminaries

In this section, we lay the groundwork for the remainder of this paper: Section 2.1 introduces utility functions and their properties, Section 2.2 states the different axioms of revealed preference, and Section 2.3 shows how graphs can be built from a given set of observations.

2.1. Basic properties of utility functions

Let us first introduce the basic ideas of revealed preference, by considering purchasing decisions and utility maximization. Specifically, consider a world with m different goods. The decision maker selects a *bundle* of goods, denoted by the $(m \times 1)$ vector $q \in \mathbb{R}^m_+$. Throughout this paper, except where noted otherwise, we assume this choice is constrained by a *linear* budget constraint. The $(1 \times m)$ vector $p \in \mathbb{R}^m_{++}$ denotes the *prices* of the goods, and b the available budget. Under the classical hypothesis of utility maximization, the choice of the decision maker is guided by a utility function $u(q)$: $\mathbb{R}^m_+ \rightarrow \mathbb{R}_+$. Thus, the decision maker selects (consciously or not) an optimal bundle q by solving the following problem, for any given price vector p and budget b.

$$\text{Maximize } u(q) \tag{1}$$

$$\text{subject to } pq \leq b. \tag{2}$$

Following standard economic theory, we assume the utility function to be concave, continuous and strictly monotone, a set of properties we capture in the following definition:

Definition 1 Well-behaved utility function. A utility function $u(q) : \mathbb{R}^m_+ \to \mathbb{R}_+$ is *well-behaved* if and only if u is concave, continuous, and strictly monotone.

Notice that in this survey, we restrict ourselves exclusively to the deterministic setting where the utility function does not depend on unobservable, random elements beyond the bundle q.

Another relevant property of a utility function is the potential uniqueness of its optima. This is formulated as follows:

Definition 2 Single-valued utility function. A utility function $u(q) : \mathbb{R}^m_+ \to \mathbb{R}_+$ is *single-valued* if and only if, for each p, b, the problem $\{\text{Maximize}\, u(q)\, \text{subject to}\, pq \leq b\}$ has a unique optimal solution q.

Of course, there are many other properties that one may want to require from a utility function; we come back to this issue in Section 4.

2.2. Preference relations and axioms of revealed preference

In the remainder of the paper, we assume that data is collected by observing, at n different points in time, the prices and quantities of all goods that are bought. This yields a dataset $S = \{(p_i, q_i)|\ i \in N\}$, where $p_i \in \mathbb{R}^m_{++}$ is the vector of prices at time i, $q_i \in \mathbb{R}^m_+$ is the bundle purchased at time i, and $N = \{1, 2, \dots, n\}$. We use the word *observation* to denote a pair (p_i, q_i), $i \in N$.

Samuelson (1938) introduced the definition of the *direct revealed preference relation* over the set of bundles.

Definition 3 Direct revealed preference relation. For any pair of observations $i, j \in N$, if $p_i q_i \geq p_i q_j$, we say that q_i is *directly revealed preferred over* q_j, and we write $q_i\, R_0\, q_j$.

The interpretation of Definition 3 is quite intuitive: indeed, note that $p_i q_i$ and $p_i q_j$ respectively express the total price of bundle q_i and bundle q_j at time i, that is, when the prices p_i apply. If the inequality $p_i q_i \geq p_i q_j$ holds, we thus observe that bundle q_i was purchased at time i *in spite of the fact that q_i was at least as expensive as q_j* at time i. The natural conclusion is that the decision maker prefers bundle q_i over q_j (otherwise, she would have bought q_j), and this is the meaning of the relation R_0.

Assume now that we wish to test the hypothesis of utility maximization. In the empirical setting, the budget available to the decision maker at time $i \in N$ is generally unobservable, but it is natural to assume that it is equal to $p_i q_i$. (As a matter of fact, if the decision maker maximizes her utility and if the utility function is monotonic, then the bundle picked at each period must exhaust the available budget, which is therefore equal to $p_i q_i$ at time i.)

We now wish to test whether the given dataset is consistent with the theory of utility maximization. For the data to be consistent with that theory, there must exist a utility function such that all purchasing decisions maximize utility under the budget constraints. We say that a utility function satisfying this requirement *rationalizes* the data, and we call it a *rationalizing utility function*.

Definition 4. Rationalizability
A dataset $S = \{(p_i, q_i)|\ i \in N\}$ is *rationalizable by a well-behaved (single-valued) utility function* if and only if there exists a well-behaved (single-valued) utility function u such that for every observation $i \in N$,

$$u(q_i) \geq u(q_j) \text{ for all } j \in N \text{ with } p_i q_i \geq p_i q_j.$$

This rationalizability concept is key in revealed preference theory, and goes back to the work of Antonelli (1886). In words, Definition 4 expresses that, at each time $i \in N$, the choice of the decision maker was rational in the sense that she picked the bundle which maximizes her utility among all (observed) bundles q_j, $j \in N$, whose total price $p_i q_j$ (at time i) was within the budget $p_i q_i$. Restricting the attention to the finite set of bundles $\{q_j | j \in N\}$ that actually have been observed in the dataset, rather than considering the infinite universe $\mathbb{R}^m_+$ of all bundles that could potentially be bought by the decision maker, will allow us to test Definition 4 in an empirical setting, as we will find out in the next sections.

In terms of the direct revealed preference relation, the utility function $u(q)$ rationalizes the data if and only if $u(q_i) \geq u(q_j)$ for all $i, j \in N$ such that $q_i\, R_0\, q_j$: in the terminology of Fishburn (1970), this means that $u(q)$ is *order-preserving* for R_0; see also Bouyssou and Pirlot (2016). Therefore, it is natural to investigate conditions on R_0 which ensure that a data set is rationalizable. This observation led Samuelson (1938) to formulate the *Weak Axiom of Revealed Preference*.

Definition 5 Weak Axiom of Revealed Preference (WARP). A dataset S satisfies WARP if and only if, for each pair of distinct bundles q_i, q_j, $i, j \in N$ with $q_i\, R_0\, q_j$, it is not the case that $q_j\, R_0\, q_i$.

WARP is the first rationalizability condition proposed in the literature. It requires the revealed preference relation to be asymmetric. The intuition behind it is simple: if the decision maker shows through her decision that she prefers bundle q_i over q_j at time i, then she cannot at another time show that she prefers q_j over q_i (assuming she behaves as a utility maximizer). In other words, WARP is a necessary condition for rationalizability by a single-valued utility function (see Section 3). On the other hand, we notice that WARP does not require the direct revealed preference relation to be transitive, so that WARP is not sufficient for rationalizability.

The work of Samuelson was further developed by Houthakker (1950), who noted that by using transitivity, the direct revealed preference relation could be extended to an indirect relation.

Definition 6 Revealed preference relation. For any sequence of observations $i_1, i_2, \dots, i_k \in N$, if $q_{i_1}\, R_0\, q_{i_2}\, R_0 \dots R_0\, q_{i_k}$, we say that q_{i_1} is *revealed preferred over* q_{i_k}, and we write $q_{i_1}\, R\, q_{i_k}$.

Using these revealed preference relations, Houthakker formulated the *Strong Axiom of Revealed Preference*.

Definition 7 Strong Axiom of Revealed Preference (. SARP)
A dataset S satisfies SARP if and only if for each pair of distinct bundles q_i, q_j, $i, j \in N$ with $q_i\, R\, q_j$, it is not the case that $q_j\, R_0\, q_i$.

In order to allow for indifference between bundles, Varian (1982) introduced the strict direct revealed preference relation, and using this relation, defined the *generalized axiom of revealed preference*, GARP.

Definition 8 Strict direct revealed preference relation. For any pair of observations $i, j \in N$, if $p_i q_i > p_i q_j$, we say that q_i is *strictly revealed preferred over* q_j, and we write $q_i\, P_0\, q_j$.

Definition 9 Generalized Axiom of Revealed Preference (. GARP)
A dataset S satisfies GARP if and only if for each pair of distinct bundles, q_i, q_j, $i, j \in N$, such that $q_i\, R\, q_j$, it is not the case that $q_j\, P_0\, q_i$.

Example 1. Consider the following small dataset consisting of four observations.

$$p_1 = (2, 2, 2) \qquad q_1 = (2, 2, 2)$$
$$p_2 = (1, 2, 4) \qquad q_2 = (4, 0, 2)$$
$$p_3 = (2, 1, 3) \qquad q_3 = (4, 4, 0)$$
$$p_4 = (4, 2, 1) \qquad q_4 = (0, 1, 4)$$

Table 1 contains the values $p_i q_j$ for $i, j = 1, \dots, 4$.

Table 1
$p_i q_j$ for $i, j = 1, \dots, 4$.

	q_1	q_2	q_3	q_4
p_1	**12**	12	16	10
p_2	13	**12**	12	18
p_3	12	14	**12**	13
p_4	14	18	24	**8**

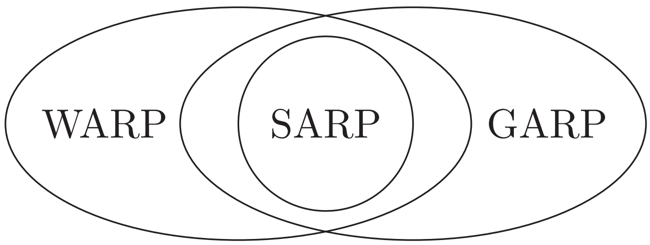

Fig. 1. Relations of the axioms of revealed preference.

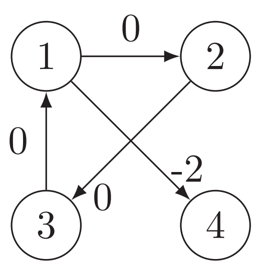

Fig. 2. A revealed preference graph.

Clearly, there are direct revealed preference relations $q_1 R_0 q_2$, $q_2 R_0 q_3$, $q_3 R_0 q_1$ and a strict direct revealed preference relation $q_1 P_0 q_4$. This dataset satisfies both WARP and GARP, but not SARP since $q_1 R q_3$ and $q_3 R_0 q_1$.

Fig. 1 illustrates the relations between the different core axioms of revealed preference theory (WARP, SARP, and GARP). Indeed, any dataset satisfying SARP satisfies both WARP and GARP, and there exist datasets not satisfying SARP that satisfy both WARP and GARP (see Example 1).

2.3. Graphs representing a dataset

We now describe how to build a directed graph that can be used to represent a dataset; this construction originates from Koo (1971). As we wil see in Section 3, such graphs are very useful tools in deciding rationalizability. Given a datset $S = \{(p_i, q_i) | \ i \in N\}$, we build a directed weighted graph $G_S = (V_S, A_S)$ as follows. For each observation $i \in N$, there is a node in V_S, i.e., $V_S := N$. Further, there is an arc from node i to node j in A_S exactly when $p_i q_i \geq p_i q_j$ and $q_i \neq q_j$ (or equivalently, when $q_i R_0 q_j$ and $q_i \neq q_j$). Observe that in G_S there is no arc between distinct observations that feature an identical bundle. Finally, the length of an arc $(i, j) \in A_S$ equals $p_i(q_j - q_i)$. Notice that this length is always nonpositive.

Example 1 Continued. The revealed preference graph corresponding to the dataset is given in Fig. 2. Notice that the direct, but not strict, revealed preference relations correspond to an arc of length 0, while the strict revealed preference relations correspond to arcs of strictly negative length.

An alternative version of this construction was proposed by Talla Nobibon et al. (2016). These authors defined a directed graph G_{R_0} which is simply the graph of the direct preference relation R_0: the node set of G_{R_0} is again N, and there is an arc from node i to node j if and only if $q_i R_0 q_j$ (including when $q_i = q_j$). For the dataset in Example 1, $G_{R_0} = G_S$ since no bundle appears twice.

3. Fundamental results

In this section, we connect the fundamentals given in Section 2, and we formulate the theorems that characterize rationalizability. Clearly, a main goal within revealed preference theory is to test whether there exists a (particular) utility function rationalizing a given dataset S.

3.1. Testing GARP

Necessary and sufficient conditions for rationalizability of a given dataset by a well-behaved utility function are given in Theorem 1.

Theorem 1. *(GARP)*
The following statements are equivalent:

1. *The dataset $S = \{(p_i, q_i) | \ i \in N\}$ is rationalizable by a well-behaved utility function $u(q)$.*
2. *There exist strictly positive numbers U_i, λ_i for $i \in N$ satisfying the system of linear inequalities*

$$U_i \leq U_j + \lambda_j p_j (q_i - q_j) \quad \forall i, j \in N. \tag{3}$$

3. *S satisfies GARP.*
4. *Each arc contained in a cycle of the graph G_S has length 0.*

The inequalities comprising system (3) are called the *Afriat Inequalities*. It is not difficult to see that system (3) can be reformulated as a linear program. Indeed, notice that multiplying a given feasible solution $(U_i, \lambda_i\!: i \in N)$ by any positive constant gives again a feasible solution; thus, one can require each of the variables to be at least equal to 1, and not just strictly positive. The equivalence of statements 1 and 2 in Theorem 1 was established by Afriat (1967b), and their equivalence with statement 3 is due to Varian (1982). Statement 4 is easily derived from the definition of GARP. Thus, Afriat (1967b) provided a linear program, formed by the Afriat Inequalities, that characterizes rationalizability by a well-behaved utility function. This allows us to conclude that GARP can be tested in polynomial time (although no polynomial time algorithms for solving linear programming problems were known at the time when Afriat published his work).

Rationalizability tests for consistency of datasets with GARP have gone through a number of stages. Diewert (1973) states another linear programming formulation. Varian's formulation of GARP (Varian, 1982) provides another algorithm for testing rationalizability. This formulation shows that rationalizability can be tested by computing the transitive closure of the direct revealed preference relation. This transitive closure yields all revealed preference relations, direct and indirect. Given the transitive closure, GARP can be tested by checking, for each pair of bundles q_i, q_j, i, $j \in N$, whether both $q_i R q_j$ and $q_j P_0 q_i$ simultaneously hold. The bottleneck in this procedure is the computation of the transitive closure. Varian suggests to use Warshall's algorithm (Warshall, 1962), which has a worst-case time complexity of $O(n^3)$; he also notes the existence of faster algorithms based on matrix multiplication, which at the time achieved $O(n^{2.74})$ complexity (Munro, 1971). By now, these algorithms have improved, the best known algorithms for general matrices having $O(n^{2.373})$ time complexity (Coppersmith & Winograd, 1990; Le Gall, 2014; Williams, 2012).

Recently, Talla Nobibon, Smeulders, and Spieksma (2015) described an algorithm with a worst-case bound of $O(n^2)$ for GARP, based on the computation of strongly connected components of the graph G_S. An alternative, simple statement of the $O(n^2)$ test is derived in Talla Nobibon et al. (2016) from the observation that a dataset S satisfies GARP if and only if $p_i q_i = p_i q_j$ for each arc (i, j) contained in a strongly connected component of G_{R_0} (see Condition 4 of Theorem 1). Shiozawa (2016) describes yet another way to test GARP in $O(n^2)$ time, using shortest path algorithms. Talla Nobibon et al. (2015) prove a lower bound on testing GARP, showing that no algorithm can exist with time complexity smaller than $O(n \log n)$.

3.2. Testing SARP

Analogously to Theorem 1, we now give a theorem that provides necessary and sufficient conditions relating to SARP.

Theorem 2. *(SARP)*
The following statements are equivalent:

1. *The dataset $S = \{(p_i, q_i)|\ i \in N\}$ is rationalizable by a well-behaved, single-valued utility function $u(q)$.*
2. *There exist strictly positive numbers U_i, λ_i for $i \in N$ satisfying the system of linear inequalities*
$$U_i < U_j + \lambda_j p_j (q_i - q_j) \quad \forall i, j \in N. \tag{4}$$
3. *S satisfies SARP.*
4. *The graph G_S is acyclic.*

Houthakker (1950), extending the work of Samuelson, introduced the formulation of SARP and proved the equivalence of statements 1 and 3. Statement 2 is an extension of Theorem 1.

Again, observe that system (4) can be cast into a linear optimization format. Using a matrix representation of the direct revealed preference relations, Koo (1963) describes a sufficient condition for consistency with SARP. Dobell (1965) is the first to describe conditions which are both necessary and sufficient. Dobell's test is based on the matrix representation of direct revealed preference relations. He proposes checking whether every square submatrix of the direct revealed preference matrix contains at least one row and one column consisting completely of elements equal to 0. Since there is an exponential number of such submatrices, this test runs in exponential time. Koo (1971) later publishes another paper where he observes that testing SARP amounts to checking whether G_S is acyclic: this can be done in $O(n^2)$ time, and is to-date the most efficient available method for testing consistency with SARP. An alternative version of this test is provided by Talla Nobibon et al. (2016). These authors observe that S satisfies SARP if and only if, within each strongly connected component of G_{R_0}, all bundles are identical. This condition can again be checked in $O(n^2)$ time by relying on Tarjan's algorithm to compute all strong components of G_{R_0} (Tarjan, 1972).

3.3. Testing WARP

For the sake of completeness, let us now state an easy result which is in fact nothing but a restatement of the definition of WARP.

Theorem 3. *(WARP)*
The following statements are equivalent:

1. *The dataset $S = \{(p_i, q_i)|\ i \in N\}$ satisfies WARP.*
2. *The graph G_S does not contain any cycle consisting of two arcs.*

As mentioned before, satisfying WARP is only a necessary condition for rationalizability by a single-valued utility function. However, in the special case where the dataset involves only two goods (i.e., $m = 2$), WARP is both a necessary and sufficient condition for rationalizability by a single-valued utility function (Little, 1949; Samuelson, 1948).

Testing WARP can be done in $O(n^2)$ time, since it is sufficient to test each pair of observations for a violation. More explicitly, after having computed the quantities $p_i q_i$ and $p_i q_j$ for all distinct $i, j \in N$, WARP can be rejected if and only if there exists a pair of distinct i, $j \in N$ such that $p_i q_i \geq p_i q_j$ and $p_j q_j \geq p_j q_i$.

Finally, let us point out that the graph characterization of GARP, SARP and WARP allows us to easily conclude (using Fig. 2) that the dataset given in Example 1 satisfies WARP (as there are no 2-cycles in G_S), satisfies GARP (as the cycle 1-2-3 has length 0), and does not satisfy SARP (as G_S is not acyclic).

Rationalizability questions are not limited to general utility functions. In the next sections, we are interested in the question whether datasets can be rationalized by utility functions of a specific form (Section 4), by collective choice processes (Section 6), or by stochastic choice processes (Section 7).

4. Other classes of utility functions and their rationalizability

Besides the basic tests discussed in the previous paragraphs, conditions and tests have been derived for testing rationalizability by various specific forms of utility functions. In this section we consider two additional classes of utility functions: utility functions that are *separable* (Section 4.1), and utility functions that are *homothetic* (Section 4.2). In addition, we assume from now on that the utility functions are *non-satiated*. This is a concept used to model the property that for every bundle q there is another bundle q' in the neighborhood of q that is preferred over q. Formally (Jehle & Reny, 2011):

Definition 10 Non-satiated utility functions. A utility function $u(\cdot)$ is non-satiated if, for each $q \in \mathbb{R}^m$ and for each $\epsilon > 0$, there exists $q' \in \mathbb{R}^m$ with $||q' - q|| \leq \epsilon$ such that $u(q') > u(q)$.

The property of non-satiatedness expresses that, in the absence of a budget constraint, no particular bundle is preferred to all other bundles. It also imposes some form of continuity to the preferences over bundles.

4.1. Separable utility functions

Separability of a utility function refers to the property that different goods in a bundle may have no joint effect on the utility of the bundle; then, goods can be regarded as independent of each other. More generally, it is often assumed that there exists a partition of the goods into R subsets such that goods from different sets do not interact. Hence, separability of a utility function is defined with respect to a given partition of the goods. More concretely, given a partition of the goods into R disjoint sets, we denote by m_j the number of goods in set j, $1 \leq j \leq R$. Any bundle of goods can then be written as $q = (q^1, \ldots, q^R)$, with $q^j \in \mathbb{R}_+^{m_j}$ denoting the vector of quantities for the goods in set j, $1 \leq j \leq R$.

There are two versions of separability: strong and weak. We first provide the definition of a strongly separable (also known as *additive*) utility function.

Definition 11 Strongly separable utility functions. A utility function $u(q)$ is strongly separable with respect to a given partition of the set of goods $\{1, 2, \ldots, m\}$ if and only if there exist well-behaved functions $f_j(q^j) : \mathbb{R}_+^{m_j} \to \mathbb{R}_+$ for each $j \in \{1, \ldots, R\}$ such that

$$u(q) = f_1(q^1) + f_2(q^2) + \cdots + f_R(q^R).$$

The case where we partition the set of goods into two subsets, i.e., the case $R = 2$, allows the following theorem due to Varian (1983):

Theorem 4. *The following statements are equivalent:*

1. *There exists a strongly separable, well-behaved, non-satiated utility function $u(f(q^1), q^2)$ rationalizing the dataset $S = \{(p_i, q_i)|\ i \in N\}$.*

2. *There exist strictly positive numbers U_i, V_i, λ_i with $i \in N$ satisfying the system of linear inequalities*

$$U_i \leq U_j + \lambda_j p_j^1 (q_i^1 - q_j^1) \ \forall i, j \in N, \tag{5}$$

$$V_i \leq V_j + \lambda_j p_j^2 (q_i^2 - q_j^2) \ \forall i, j \in N. \tag{6}$$

Varian (1983) also gives a linear programming formulation for arbitrary R, allowing for a polynomial-time test of rationalizability by a strongly separable utility function.

A weaker version of separability occurs when the utilities of the different sub-bundles are not necessarily summed to obtain the total utility; weak separability rather assumes that there exists a function, denoted u', that takes as input the utilities of the individual groups of goods, and translates these into a total utility.

Definition 12 Weakly separable utility functions. A utility function $u(q)$ is weakly separable with respect to $q^1, \ldots, q^{R-1}$ if and only if there exist functions $f_j(q^j) : \mathbb{R}_+^{m_j} \to \mathbb{R}_+$ for each $j \in \{1, \ldots, R-1\}$ and a function $u'(x_1, \ldots, x_{R-1}, q^R)$ such that $u(q) = u'(f_1(q^1), \ldots, f_{R-1}(q^{R-1}), q^R)$.

Following his paper on general utility functions, Afriat also wrote an unpublished work on separable utility functions (Afriat, 1967a). Varian (1983) built further on this, giving a non-linear system of inequalities, reproduced below in Theorem 5, for which the existence of a solution is a necessary and sufficient condition for rationalizability by a well-behaved, weakly separable utility function with $R = 2$ sets of goods.

Theorem 5. *The following statements are equivalent.*

1. *There exists a weakly separable, well-behaved, non-satiated utility function $u(f(q^1), q^2)$ rationalizing the dataset $S = \{(p_i, q_i) | \ i \in N\}$.*
2. *There exist strictly positive numbers U_i, V_i, λ_i, μ_i for $i \in N$ satisfying the system of non-linear inequalities*

$$U_i \leq U_j + \lambda_j p_j^2 (q_i^2 - q_j^2) + (\lambda_j / \mu_j)(V_i - V_j) \ \forall i, j \in N, \tag{7}$$

$$V_i \leq V_j + \mu_j p_j^1 (q_i^1 - q_j^1) \ \forall i, j \in N. \tag{8}$$

Diewert and Parkan (1985) extend this result to multiple separable subsets. Cherchye, Demuynck, De Rock, and Hjertstrand (2015) prove that testing rationalizability by a weakly separable utility function is NP-HARD even for $R = 2$. They also provide an integer programming formulation which is equivalent to (7) and (8). Several heuristic approaches have been formulated for testing weak separability. Varian attempts to overcome the computational difficulties by finding a solution to the linear part of the system of inequalities and then fixing variables based on this solution, which linearizes the remainder of the inequalities. This implementation can be too restrictive, as the variables are usually fixed with values making the system infeasible, even if a solution exists, as shown by Barnett and Choi (1989). Fleissig and Whitney (2003) take a similar approach, but improve on it by fixing variables with values that are more likely to allow solutions to the rest of the system of equalities. Exact tests of (adaptations of) Varian's inequalities are described in Swofford and Whitney (1994) and Fleissig and Whitney (2008). Both use non-linear programming packages to find solutions and are limited in the size of datasets they can handle. Computational results in Cherchye et al. (2015) suggest that the integer programming approach is feasible for moderately sized datasets. Hjertstrand, Swofford, and Whitney (2016) use this approach in an application testing separability of consumption, leisure and money. When dropping the concavity assumption, the rationalizability problem remains NP-HARD, even if the dataset is limited to 9 goods (Echenique, 2014). Quah (2014) provides an algorithm for testing separable utility functions without the concavity assumption. Swofford and Whitney (1994) modify (7) and (8) to account for consumers needing time to adjust their spending.

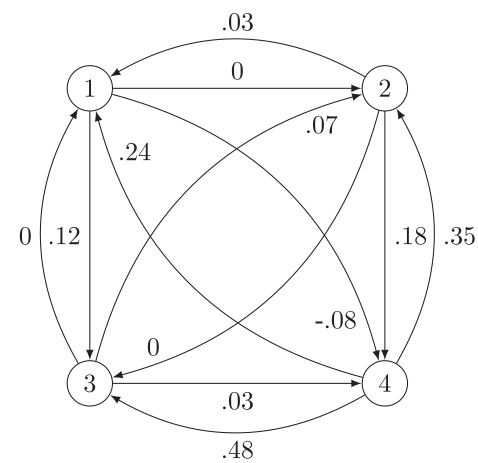

Fig. 3. A revealed preference graph for testing homotheticity.

4.2. Homothetic utility functions

Another class of utility functions of interest are the *homothetic utility functions*. Their definition is based on the concept of a homogenous function.

Definition 13 Homogenous functions. A function $f(\cdot)$ is homogenous when $f(\lambda q) = \lambda f(q)$, for each $q \in \mathbb{R}^m$ and for each $\lambda \in \mathbb{R}$.

Definition 14 Homothetic utility functions. A utility function $u(\cdot)$ is homothetic when there exist a homogenous function f and a monotonic function ℓ such that $u(q) = \ell(f(q))$ for each $q \in \mathbb{R}^m$.

In effect, if u is homothetic and if $u(q_i) \geq u(q_j)$ for two bundles q_i, q_j, then for any constant $\alpha > 0$, $u(\alpha q_i) \geq u(\alpha q_j)$. Theorem 6 gives necessary and sufficient conditions for rationalizability of a dataset by a homothetic utility function. Notice that for tests of homothetic utility functions described in the theorem, we assume the price vectors are normalized so that $p_i q_i = 1$ for all $i \in N$. One of these conditions is based on the following graph $H = (V_S, A_S)$ (whose construction is in the spirit of the construction described in Section 2.3). For each observation $i \in N$, there is a node in V_S, i.e., $V_S := N$. Further, for each ordered pair of observations (i, j), there is an arc of length $\log(p_i q_j)$ between the corresponding nodes. Fig. 3 shows a graph to test homotheticity for the dataset given in Example 1.

Theorem 6. *The following statements are equivalent:*

1. *There exists a non-satiated homothetic utility function $u(\cdot)$ rationalizing the dataset $S = \{(p_i, q_i) | \ p_i q_i = 1, \ \forall i \in N\}$.*
2. *There exist strictly positive numbers U_i for $i \in N$ satisfying the inequalities*

$$U_i \leq U_j p_j q_i \ \forall i, j \in N. \tag{9}$$

3. *For all distinct choices of observations $(i_1, i_2, \ldots, i_k)$, we have*

$$(p_{i_1} q_{i_2})(p_{i_2} q_{i_3}) \ldots (p_{i_k} q_{i_1}) \geq 1. \tag{10}$$

4. *The graph H_S does not contain a cycle of negative length.*

The equivalence of statements 1, 2 and 3 was proven by Afriat (1972, 1981). Based on statement 4, Varian (1983) proposes a combinatorial test which can be implemented in $O(n^3)$ time.

Table 2
Complexity results for testing rationalizability by utility functions of specific forms.

Type of utility function	Type of test	Time complexity
General	Graph test	$O(n^2)$
Single-valued	Graph test	$O(n^2)$
Strongly separable	System of linear ineq.	Polynomial
Weakly separable	System of non-linear ineq.	NP-HARD
Homothetic	Graph test	$O(n^3)$
Homothetic and separable	System of non-linear ineq.	Open

Varian (1983) also provides a test for homothetic, separable utility functions, which is again a difficult-to-solve system of non-linear inequalities. Finally, utility maximization in case of rationing (i.e., when there are additional linear constraints on the bundles which can be bought, on top of the budget constraint) is also handled by Varian. He provides a linear system of inequalities whose feasibility is a necessary and sufficient condition for rationalizability.

In summary, various forms of utility functions are usually associated with a system of inequalities, for which the existence of a solution is a necessary and sufficient condition for rationalizability by such a utility function. The difficulty of these rationalizability tests crucially depends on whether the systems are linear or non-linear. General, single-valued and strongly separable utility functions are easy to rationalize, as their associated systems of inequalities are linear. The same holds true for utility maximization by a general utility function under rationing constraints. For general and single-valued utility functions, more straightforward tests have been developed. A polynomial test also exists for rationalizability by a homothetic utility function. On the other hand, for those utility functions associated with non-linear systems of inequalities, that is, weakly separable and homothetic separable functions, no efficient tests are known. For weakly separable utility, formal NP-HARDNESS results exist. For homothetic separable functions, the complexity question remains open. Varian (1982, 1983) provides a way to construct consistent utility functions for all of these settings. Table 2 summarizes these results.

To complete our overview on rationalizability by general utility functions, we mention some recent work on indivisible goods and non-linear budget sets. More precisely, these are settings where the optimization problems (1) and (2) are further constrained by the conditions that (i) some components of q are integral, and (ii) the budget constraint is non-linear (e.g., in the presence of quantity discounts), and/or there are multiple budget constraints. Forges and Minelli (2009) give a revealed preference characterization for non-linear budgets, for which GARP is a sufficient and necessary condition for rationalizability by an increasing and continuous utility function. Cherchye, Demuynck, and De Rock (2014) give conditions for rationalizability by an increasing, concave and continuous utility function for the setting with non-linear budgets. They note that, together with the results by Forges and Minelli, this allows for tests of the concavity of utility functions which are not possible in the setting with linear budgets. Computationally there is no obvious easy way to test the conditions laid out by Cherchye et al. in general. However, they show that if the budgets can be represented by a finite union of polyhedral convex sets, a system of linear inequalities provides conditions for rationalizability. Fujishige and Yang (2012) and Polisson and Quah (2013) extend the revealed preference results to the case with indivisible goods. They find that GARP is a necessary and sufficient test for rationalizability, given a suitable adaptation of the revealed preference relations for their setting. Cosaert and Demuynck (2015) look at choice sets which are non-linear and have a finite number of choice alternatives. They provide revealed preference characterizations for weakly monotone, strongly monotone, weakly monotone and concave, and strongly monotone and concave utility functions, all of which are easy to test, either by some variant of GARP or a system of linear inequalities.

5. Goodness-of-fit and power measures

An often cited limitation of rationalizability tests is that they are binary tests: either the dataset is rationalizable or it is not. Thus, when violations of rationalizability conditions are found, there is no indication of their severity. Likewise, when the rationalizability conditions are satisfied, this could be because the choices faced by the decision maker make it unlikely that violations would occur. To refine this yes/no verdict inherent to rationalizability, so-called goodness-of-fit measures and power measures have been proposed in the literature. Goodness-of-fit measures (Section 5.1) quantify the severity of violations, while power measures (Section 5.2) indicate how far the choices are from violating rationalizability conditions.

5.1. Goodness-of-fit measures

A first class of goodness-of-fit measures is based on the systems of inequalities which are used to establish rationalizability of many different forms of utility functions (see Section 3). Slack variables are added to these systems, so as to relax the constraints on the data. An optimization problem can then be defined, for which the objective function is the minimization of some appropriate function of the slack variables, such as their sum, under the constraint that the system of equalities is satisfied. The goodness-of-fit measure is then equal to the value of the optimal solution of this optimization problem. Such an approach was first described by Diewert (1973) and has since been used in a number of different papers for various forms of utility functions (see Diewert & Parkan, 1985; Fleissig & Whitney, 2005; Fleissig & Whitney, 2008 for weak separability, Fleissig and Whitney (2007) for additive separability). Computing the goodness-of-fit measure is easy if the system of inequalities is linear, which is the case for general utility functions and additive separable utility functions. In the case of non-linear systems of inequalities, minimizing the sum of the slack variables is at least as hard as finding a solution to the system without slack variables. Since this is already NP-HARD for weakly separable utility functions, the hardness result remains valid for these goodness-of-fit measures.

A second class of goodness-of-fit measures is due to Afriat (1973), and is based on strengthening the revealed preference relations. In this case, revealed preference relations are assumed to hold if the difference in price between the chosen bundle and another affordable bundle is big enough. This is done by introducing *efficiency indices* $0 \leq e_i \leq 1$ for each observation $i \in N$, and defining the revealed preference relation $R_0(e_1, \ldots, e_n)$ as follows:

$$\text{for all } i, j \in N, \text{ if } e_i p_i q_i \geq p_i q_j, \text{ then } q_i R_0(e_1, \ldots, e_n)\, q_j. \tag{11}$$

Obviously, when $e_i = 1$, conditions (11) are the same revealed preference relations as in Definition 3; when $e_i < 1$, condition (11) can be interpreted as defining a revealed preference relation between two bundles for which the price difference exceeds a certain fraction of the budget. As a result, there will be fewer revealed preference relations, and axioms such as WARP, SARP and GARP will be easier to satisfy. A goodness-of-fit measure is then the maximum value of the sum of the e_i values, under the constraint that a given axiom of revealed preference is satisfied by $R_0(e_1, \ldots, e_n)$. Three different goodness-of-fit indices based on this idea have been respectively described by Afriat (1973), Varian (1990) and Houtman and Maks (1985). Of these three, Afriat's index is the simplest, as it constrains the e_i values to be equal for every observation ($e_1 = e_2 = \cdots = e_n$). Afriat's index can be computed in polynomial time (see Smeulders, Spieksma, Cherchye, & De Rock, 2014), although for a long time the only published algorithm was an approximation algorithm due to Varian (1990). Varian's index, in contrast, allows the e_i values to differ between observations. This makes computation less straightforward and the computation of this index was thus perceived to be hard (as confirmed by Smeulders et al. (2014) who showed that computing Varian's index is NP-HARD). This led to work on heuristic algorithms for computing Varian's index by Varian (1990), Tsur (1989), and more recently by Alcantud, Matos, and Palmero (2010). Finally, Houtman and Maks (1985) proposed to constrain the e_i values to be either 0 or 1. In effect, maximizing the sum of the e_i's then amounts to removing the minimum number of observations so that the remaining dataset is rationalizable. Houtman and Maks established a link between the feedback vertex set problem (known to be NP-HARD) and their index, thus informally showing its difficulty; see Hjertstrand and Heufer (2015) for two methods computing the Houtman–Maks index. The complexity of computing all three of the above indices is addressed by Smeulders et al. (2014), who provide polynomial time algorithms for Afriat's index for various axioms of revealed preference, and establish NP-hardness of Varian's index, and of the Houtman–Maks index. Even stronger, it is shown that no constant-factor approximation algorithms running in polynomial time exist for these indices unless P = NP. Boodaghians and Vetta (2015) strengthen these hardness results, by showing that computing the Houtman–Maks index is already NP-HARD for datasets with only 3 goods.

A third approach to the definition of goodness-of-fit measures was introduced by Varian (1985). When a dataset fails to satisfy the rationalizability conditions, the goal is here to find a dataset which does satisfy the conditions and is only minimally different from the observed dataset. The problem of finding these minimally different rationalizable datasets can be formulated as a non-linear optimization problem, which, in general, is hard to solve. To avoid solving large scale non-linear problems, De Peretti (2005) approaches this problem with an iterative procedure. Working on GARP, his algorithm tackles violations one at a time, also perturbing only one observation at a time. If a preference cycle exists between two bundles of goods q_i and q_j, $i, j \in N$, he computes the minimal perturbation necessary to remove the violation both for the case in which $q_i R_0 q_j$ (in which case q_i is perturbed) and for the case in which $q_j R_0 q_i$ (in which case q_j is perturbed). The smallest of the two perturbations is then used to update the dataset, and the new dataset is checked again for GARP violations. While this algorithm does not guarantee an optimal solution, it allows handling large datasets, especially if the number of violations is small.

A number of recent papers introduce new goodness-of-fit measures, thus showing continued interest in this topic. Echenique et al. (2011) define the mean and median money pump indices. In their paper, the severity of violations of rationality is measured by the amount of money which an arbitrageur could extract from the decision maker by exploiting her irrational choices. This is reflected by a money pump index for every violation of rationality. Echenique et al. propose to calculate the money pump index of the mean and median violation as measures of the irrationality of the decision maker. Computing these measures is NP-HARD, as shown in Smeulders, Cherchye, De Rock, and Spieksma (2013). In the latter paper, it is also shown that computing the money pump index for the most and least severe violations can be done in polynomial time. Furthermore, Apesteguia and Ballester (2015) introduce the minimal swaps index. Informally, the swaps index of a given preference ordering over the alternatives is calculated by counting how many better alternatives (according to the preference order) were not chosen over all choice situations. The minimal swaps index is then the swaps index of the preference order for which this index is minimal. Apesteguia and Ballester show that computing the minimal swaps index is equivalent to the NP-HARD linear ordering problem. Finally, Dean and Martin (2016) define the minimum cost index. This index is the minimum cost of removing revealed preference relations, such that the remaining relations induce no violations. The cost of removing violations is weighted by the price difference of the considered bundles. Dean and Martin show that computing this index is NP-HARD by a reduction from the set covering problem.

5.2. Power measures

Power measures were first introduced by Bronars (1987), with the following motivation. Consider a test that allows us to determine whether the observations in a dataset are coherent with the choices of a utility-maximizing decision maker. If the outcome of the test is positive for most datasets, *including those where choices were not made so as to maximize a utility function,* then obviously the test is not good at discriminating between utility maximizing behavior and alternative behaviors. Power measures are numerical values indicating to what extent a test is able to discriminate between samples coming from a rational or from an irrational decision maker.

Bronars (1987) proposes to use random choices as an alternative model of behavior. The likelihood of this alternative model satisfying the rationalizability conditions (that is, passing the test) is determined by Monte Carlo simulation. The higher this likelihood, the lower the power of the test. Andreoni and Miller (2002) use a similar approach: they generate synthetic datasets by bootstrapping from observed choices, and use these alternative datasets to establish the power of their test.

Bronars's Monte Carlo approach has also been applied to goodness-of-fit measures. The value of a goodness-of-fit measure is hard to interpret without context. There is no natural level which, if crossed, indicates a large deviation from rational behavior. Furthermore, the values of goodness-of-fit indices which point to large deviations may vary from dataset to dataset, as the choices faced by a decision maker may or may not allow large violations of rationalizability. One way to establish what values are significant, is to generate random datasets by a Monte Carlo approach and to calculate their goodness-of-fit measures. This yields a distribution of the values of goodness-of-fit measures for datasets of random choices. It can then be checked whether the goodness-of-fit measures computed for the actual decision makers are significantly different. Examples of this approach are found in Choi, Fisman, Gale, and Kariv (2007) and Heufer (2012). As this framework requires a large number of computations of the goodness-of-fit measures, there is a strong incentive to use efficient algorithms and to favor measures which are easy to calculate.

Beatty and Crawford (2011) propose to evaluate the power of a test by calculating the proportion of possible choices which would pass the test. Andreoni, Gillen, and Harbaugh (2013) give an overview of power measures and introduce a number of new power measures themselves. The measures they introduce are adaptations of goodness-of-fit measures. For example, they introduce a jittering index, which is the minimum perturbation of the data such that the rationalizability conditions are no longer satisfied, in line with the work of Varian (1985). They also introduce an Afriat Power Index, which is the converse of Afriat's goodness-of-fit measure; that is, instead of considering the maximum value of $e \leq 1$ in (11) such that the dataset satisfies the considered axiom of revealed preference, they propose to determine the minimum value of $e \geq 1$ such that the dataset does not satisfy the conditions.

6. Collective choices

In the preceding sections, datasets are analyzed as if a single person buys or chooses goods, so as to maximize her own utility function. However, in many cases purchasing decisions are observed at the household level that consists of multiple decision makers. The choices that result from collective decision making may appear irrational, even if all individual decision makers have rational preferences. For example, Arrow's impossibility theorem (Arrow, 1950) shows that for non-dictatorial, unanimous preference aggregation functions, independence of irrelevant alternatives cannot be guaranteed. As a result, the group can exhibit choice reversals if more choice alternatives are added. Moreover, a group can use different choice mechanisms at different times, giving more or less power to different group members, also leading to choices that appear irrational. Analyzing datasets resulting from collective choices thus calls for collective models, which account for individually rational household members, and in addition, some decision process for splitting up the budget. Example 2 shows how the joint purchases of two rational decision makers can appear irrational when they are analyzed as if there was a unique decision maker.

Example 2. Consider the following dataset with 2 periods and 3 goods.

$$p_1 = (3, 2, 1) \qquad q_1 = (5, 4, 7) \tag{12}$$

$$p_2 = (2, 3, 1) \qquad q_2 = (3, 5, 9) \tag{13}$$

Then, bundle 1 would be strictly revealed preferred over bundle 2, since $p_1 q_1 = 30 > 28 = p_1 q_2$. Likewise, bundle 2 would be strictly revealed preferred over bundle 1, since $p_2 q_2 = 30 > 29 = p_2 q_1$. The dataset thus does not satisfy GARP. However, consider the following datasets.

$$p_1 = (3, 2, 1) \qquad q_1^1 = (5, 0, 0)$$
$$p_2 = (2, 3, 1) \qquad q_2^1 = (3, 0, 0)$$

and

$$p_1 = (3, 2, 1) \qquad q_1^2 = (0, 4, 7)$$
$$p_2 = (2, 3, 1) \qquad q_2^2 = (0, 5, 9)$$

It is clear that both of these satisfy GARP, since for the first dataset $q_1^1 > q_2^1$, and for the second dataset $q_2^2 > q_1^2$. Furthermore, notice that $q_1 = q_1^1 + q_1^2$ and $q_2 = q_2^1 + q_2^2$. The datasets (12) and (13) thus represent the joint purchases of two rational decision makers.

The initial contributions in revealed preference theory dealing with collective choice are published by Chiappori (1988), for the so-called *labor supply* setting. This setting corresponds to a situation in which there are two goods, namely leisure time and aggregated consumption, which are observed for each member in the household. Also, we assume that the household consists of two decision makers. The behavior of this household is then rationalizable if the consumption can be split up so that the resulting individual datasets of leisure and consumption are rationalizable for all individual household members. Chiappori provides conditions for rationalizability, both for the cases with and without externalities of private consumption. To model the labor supply setting in the collective choice model, we use a dataset of the form $S = \{(w_i^1, w_i^2, L_i^1, L_i^2, C_i)|\ i \in N\}$, with w_i^1 and w_i^2 corresponding to the wages of household members 1 and 2, with L_i^1 and L_i^2 corresponding to their respective leisure time, and with C_i denoting the level of (collective) consumption in the household ($i \in N$). Notice that, since wages can be seen as the price of leisure time, and there is a unit price for aggregated consumption, we can write $p_i = (w_i^1, \mathbf{1})$ and $q_i = (L_i^1, fC_i)$ (for some fraction $0 \leq f \leq 1$). Hence, the dataset S can still be seen as a set of observations consisting of price vectors and bundles.

Theorem 7. *(Chiappori's Theorem for collective rationalization by egoistical agents)*

The following statements are equivalent.

1. *There exists a pair of concave, monotonic, continuous non-satiated utility functions which provide a collective rationalization by egoistical agents.*
2. *There exist numbers Z_i with $0 \leq Z_i \leq C_i$ such that the following (equivalent) conditions are satisfied.*
   (a) *The datasets $\{(w_i^1, \mathbf{1}), (L_i^1, Z_i)|\ i \in N\}$ and $\{(w_i^2, \mathbf{1}), (L_i^2, C_i - Z_i)|\ i \in N\}$ both satisfy SARP.*
   (b) *There exist strictly positive numbers $U_i^1, U_i^2, \lambda_i, \mu_i$ for $i \in N$ satisfying the non-linear inequalities*

$$U_i^1 \leq U_j^1 + \lambda_j w_j^1 (L_i^1 - L_j^1) + \lambda_j (Z_i - Z_j) \qquad \forall i, j \in N,$$
$$U_i^2 \leq U_j^2 + \mu_j w_j^2 (L_i^2 - L_j^2) + \mu_j (C_i - Z_i - C_j - Z_j) \qquad \forall i, j \in N,$$

with equality holding in the first (respectively, the second) inequality only if $L_i^1 = L_j^1$ and $Z_i = Z_j$ (respectively, $L_i^2 = L_j^2$ and $Z_i = Z_j$).

Theorem 7 states Chiappori's result for collective rationalization by egoistical agents. (The agents are egoistical in the sense that they each spend their own personal wages, so that the observed consumption is just the sum of the individual ones.) No straightforward method is included in the paper to test the first condition; the second condition requires solving a system of non-linear inequalities. Similar conditions hold for the case with externalities. Snyder (2000) provides a reformulation of Chiappori's conditions for two periods and uses it in empirical tests. Thanks to the limit on the number of periods, this test is very easy: it requires solving four small linear systems of inequalities. Cherchye, De Rock, and Vermeulen (2011) depart from the labor supply setting by formulating a collective model with an arbitrary number of goods. In their model, each specific good is known to be either publicly or privately consumed. Given this information, rationalizability is tested by checking whether there exists a split of prices (for public goods) or quantities (for private goods), such that the dataset of personalized prices and quantities for each household member satisfies GARP. Cherchye et al. (2011) provide an integer programming formulation to test their model. Talla Nobibon et al. (2016) provide a large number of practical and theoretical computational results for this problem. First, they prove it is NP-HARD. Furthermore, they describe a more compact integer programming formulation, and provide a simulated annealing based metaheuristic. They compare the computational results with these different integer programming formulations and heuristics; they observe that the heuristic approach is capable of tackling larger datasets and seldom fails to find a feasible split when one exists. Smeulders, Cherchye, De Rock, Spieksma, and Talla Nobibon (2015) give further hardness results for a collective version of WARP: they find that the problem remains NP-HARD when testing for transitivity is dropped. All hardness results for these problems assume that the number of goods is not fixed a priori. It remains an open question whether the problems become easy for a small, fixed number of goods. In particular, the labor supply setting only requires one good to be partitioned over members of the household.

The work by Chiappori is generalized by Cherchye, De Rock, and Vermeulen (2007). Leaving the labor supply setting, they provide conditions for an arbitrary number of goods and without any prior allocation of goods, as was the case with leisure time in Chiappori's work. Cherchye et al. (2007) derive separate necessary and sufficient conditions for collective rationalizability by concave utility functions. In a later paper, Cherchye, De Rock, and Vermeulen

(2010) show that the necessary condition given in their earlier work is both necessary and sufficient, when dropping the assumption of concave utility functions. However, testing this condition is NP-HARD, as shown by Talla Nobibon and Spieksma (2010). Due to the hardness of rationalizability in collective settings, a number of papers have appeared on how to test this problem. An integer programming formulation is given by Cherchye, De Rock, Sabbe, and Vermeulen (2008) and an enumerative approach is provided by Cherchye, De Rock, and Vermeulen (2009). Talla Nobibon, Cherchye, De Rock, Sabbe, and Spieksma (2011) take a different approach and propose a heuristic algorithm. The goal of this algorithm is to quickly test whether the rationalizability conditions are satisfied. If this heuristic cannot prove that the conditions are satisfied, then an exact test is used. Using this heuristic pre-test, many computationally demanding exact tests can be avoided. Deb (2010) strengthens the hardness results by proving that a special case of this problem, the situation dependent dictatorship setting, is also NP-HARD. In this setting, the household decision process is such that each purchasing decision is made by a single household member, called the dictator. At different points in time, different household members can assume the role of the dictator; the goal is thus to partition the observations into datasets, so that each dataset is consistent with (unitary) GARP. Crawford and Pendakur (2013) also consider this problem in the context of preference heterogeneity, and provide algorithms for computing upper and lower bounds on the number of 'dictators'. Cosaert (2017) links this to the problem of computing the chromatic number of a graph. Furthermore, Cosaert formulates an integer program to partition the observations into sets, so that the observed characteristics within each set are as homogenous as possible. Smeulders et al. (2015) give further hardness results for a collective version of WARP: they find that dropping transitivity makes the test easy for households of two members, but the problem remains open for three or more members.

7. Revealed stochastic preference

In the previous sections, we have looked at methods that decide whether a set of observations can be rationalized by one or more decision makers, using different forms of utility functions, or different ways in which the choice process can be split over several decision makers. However, we assumed that utility functions and preferences are fully deterministic. As a result, if a choice situation repeats itself, we expect that the decision maker always chooses the same alternative. However, it is commonly observed in experiments on choice behavior that if a person is given the same choice situation multiple times, her decision may change. One possible way of explaining this behavior is by stochastic preferences, as pioneered by Block and Marschak (1960). Theories of stochastic preferences posit that, while at any point in time a decision maker has a preference ordering over all alternatives, these preferences are not constant over time and may fluctuate randomly. An observed behavior is rationalizable by stochastic preferences if and only if there exists a set of utility functions and a probability distribution over these utility functions, such that the frequency with which an alternative is chosen in any given choice situation is equal to the probability that this alternative has the highest utility in that situation. We note that many results on stochastic preferences are established for the case of finite choice sets, as opposed to the consumption setting, where there exists an infinite number of bundles that can be bought for a given expenditure level and prices. For an overview, we refer to McFadden (2005).

A very general result was established by McFadden and Richter (1990), namely, the *axiom of revealed stochastic preference* (ARSP), which states a necessary and sufficient condition for rationalizability of choice probabilities by stochastic preferences. The generality of this axiom allows it to be used for any form of choice situation, and all classes of decision rules. Besides the axiom, McFadden and Richter also provided a system of linear inequalities whose feasibility is a necessary and sufficient condition for rationalizability. Neither of these characterizations can be easily operationalized, since ARSP places a condition on every possible subset of observations, so that the resulting number of conditions is exponential in the number of observations. Furthermore, each condition requires finding a decision rule among all allowed decision rules which maximizes some function, and this can in itself be an NP-HARD problem (for example when the class of decision rules being tested are based on linear preference orders, this means solving an NP-HARD linear ordering problem; Karp (1972)). The linear system of inequalities, on the other hand, contains one variable for every possible decision rule within a class of decision rules, a number which is often exponential in the number of choice alternatives.

For the setting of consumer purchases (and thus infinite choice sets), Bandyopadhyay, Dasgupta, and Pattanaik (1999) formulate the *weak axiom of stochastic revealed preference* (WARSP). This axiom provides a necessary condition for rationalizability by stochastic preferences. Analogously to WARP, WARSP compares pairs of choice situations. Since the condition placed on these pairs is easy to test, WARSP allows for a polynomial time test. Heufer (2011) and Kawaguchi (2016) build further on this work. Heufer provides a sufficient condition for rationalizability in terms of stochastic preferences. Kawaguchi (2016) proposes the *strong axiom of revealed stochastic preference* (SARSP), a necessary condition for rationalizability by stochastic preferences. Both of these conditions seem difficult to test, requiring in the case of Heufer a feasible solution to a linear program with an exponential number of constraints and variables. Kawaguchi's SARSP likewise requires checking an exponential number of inequalities. Despite these challenges, Kitamura and Stoye (2014) develop a test which can be used to test rationalizability by stochastic preferences on consumption data, though for relatively small datasets. A key element in their approach is discretizing the dataset, so as to return to a setting with a finite number of choice options.

8. Conclusion

In this final section, let us summarize our discussion, and outline perspectives regarding possible future developments in the field. It is indisputable that revealed preference theory has established itself as an important tool in economics. On the other hand, testing revealed preference axioms on large datasets gives rise to numerous algorithmic challenges that should appeal to the operations researcher community. While a thorough understanding of individual rational choice, as it relates to revealed preference, has been achieved, we see (at least) three research directions emerging:

1. Economists are increasingly extending the revealed preference setting to more complex theories of choice behavior, such as collective decision making, or non-deterministic choices. The testing problems emerging in these cases are likewise more complex. Much work, both theoretical and algorithmically remains to be done in this area.
2. Many complexity hardness results have been established under the assumption that the number of goods can be arbitrarily large, as opposed to assuming that this number is limited and fixed (e.g., $m = 2$ or $m = 3$). We have mentioned in this survey a few results that hold when the number of goods is fixed, but many questions remain open in this direction. Beyond its theoretical interest, this setting has practical relevance, since in many empirical studies the number of goods is quite small, or

goods are aggregated into a limited number of classes. Tests that are difficult in general may turn out to be polynomially computable in these cases.

3. The relevance of efficient revealed preference tests for large datasets (see Section 1.1) continues to increase due to the ever growing size of available datasets. Better algorithms, both heuristic and exact, are required in order to be able to cope with this phenomenon. Thus, we need to further increase our understanding of the achievable running times for different versions of the rationalizability question.

Answering these questions will not only reveal the inherent difficulty of testing rationalizability of a given dataset by a utility function from a particular class, it will also shed light on the incentives and properties of human behavior.

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Economic Theory

Exposita Notes

Two new proofs of Afriat's theorem

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Summary. We provide two new, simple proofs of Afriat's celebrated theorem stating that a finite set of price-quantity observations is consistent with utility maximization if, and only if, the observations satisfy a variation of the Strong Axiom of Revealed Preference known as the Generalized Axiom of Revealed Preference

Keywords and Phrases: Afriat's theorem, SARP, GARP.

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1 Introduction

The neoclassical theory of demand supposes that a consumer, facing a price vector $p \in \mathbb{R}^\ell_{++}$ and with income $I > 0$, chooses his demand bundle $x \in \mathbb{R}^l_+$ to maximize some utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ over his budget set $B(p, I) := \{x \in \mathbb{R}^\ell_+ : p \cdot x \leq I\}$. We assume we have been presented with a finite data set $D := \{(p_i, x_i) : i \in N\}$, where $N := \{1, 2, \ldots, n\}$, of price vectors $p_i \in \mathbb{R}^\ell_{++}$ and corresponding demand vectors $x_i \in \mathbb{R}^\ell_+$. The basic question raised by Afriat is whether this data set is consistent with the maximization of a locally non-satiated utility function u in the sense that for each $i \in N$, x_i maximizes u over $B(p_i, p_i \cdot x_i)$. A locally non-satiated utility function is one for which every neighborhood of a commodity bundle contains another bundle with a higher utility. With such a utility function the consumer will have spent all his income, so that we can use $p_i \cdot x_i$ as the income for situation i.

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If the set of price and quantity observations is derived from utility maximization it will surely satisfy the variation of the Strong Axiom of Revealed Preference, known as the Generalized Axiom of Revealed Preference, which states that, for any list $(x_1, p_1), \ldots, (x_n, p_n)$ with the property that

$$p_j \cdot x_{j+1} \leq p_j \cdot x_j, \text{ for all } j \leq n-1,$$

we must have $p_n \cdot x_1 \geq p_n \cdot x_n$.[1]

The argument for the Generalized Axiom is straightforward. If $p_j \cdot x_{j+1} \leq p_j \cdot x_j$ then x_{j+1} could have been purchased at prices p_j. Since x_{j+1} was not purchased it cannot be strictly preferred to x_j so that $x_j \succsim x_{j+1}$. The entire sequence of inequalities therefore implies that $x_1 \succsim x_n$. If, on the other hand, $p_n \cdot x_1 < p_n \cdot x_n$ and the utility function is locally non-satiated, we could find a commodity bundle ξ close to x_1 with $p_n \cdot \xi < p_n \cdot x_n$ and $\xi \succ x_n$, violating the assumption that x_n maximizes utility at prices p_n and income $p_n \cdot x_n$.

The Generalized Axiom may be stated in a slightly different fashion which is more appropriate for our needs. If the inequalities

$$\begin{aligned} p_j \cdot x_{j+1} &\leq p_j \cdot x_j, \text{ hold for all } j \leq n-1 \text{ and if} \\ p_n \cdot x_1 &\leq p_n \cdot x_n \text{ as well,} \end{aligned}$$

then we must have $p_n \cdot x_1 = p_n \cdot x_n$. But in this form there is no distinction between the last observation and any of the other observations, so that

$$p_j \cdot x_{j+1} = p_j \cdot x_j$$

holds for all j. This is the variation of the Strong Axiom which we shall adopt, not only for the full set of n observations but for any ordered subset as well.

Definition 1 *We say that the observations satisfy the Generalized Axiom of Revealed Preference (GARP) if for every ordered subset $\{i, j, k, \ldots, r\} \subset N$ with*

$$\begin{gathered} p_i \cdot x_j \leq p_i \cdot x_i \\ p_j \cdot x_k \leq p_j \cdot x_j \\ \vdots \\ p_r \cdot x_i \leq p_r \cdot x_r \end{gathered}$$

it must be true that each inequality is, in fact, an equality.

[1] There is a great variety of terminology associated with the concept of revealed preference. The original definition offered by Samuelson [4], now known as the Weak Axiom of Revealed Preference (WARP), was thought by the author to be sufficient to recover a utility function generating the data. Houthakker's definition of the Strong Axiom (SARP) [3] provided the additional conditions necessary for recovery. But Houthakker's statement of the Strong Axiom is motivated by a single valued demand function rather than a finite list of observations and is, as a consequence, somewhat awkward. Afriat [1] used the terminology Cyclical Consistency (CC) for the simpler concept of the current paper. Cyclical Consistency is identical with the Generalized Axiom of Revealed Preference (GARP) introduced by Varian [5]. This does not exhaust the list of variations in terminology.
We have chosen to use the term GARP rather than Cyclical Consistency. Our purpose is to use a definition in which the phrase "Revealed Preference" actually appears rather than the earlier, equivalent terminology used by Afriat.

From the data set we can compute the square matrix A of order n defined by

$$a_{ij} := p_i \cdot (x_j - x_i) \text{ for all } i, j \in N.$$

Hence, a_{ij} negative means that x_i is revealed preferred to x_j. In this more condensed notation, the observations satisfy the Generalized Axiom if for every chain $\{i, j, k, \dots, r\} \subset N$, $a_{ij} \leq 0$, $a_{jk} \leq 0, \dots, a_{ri} \leq 0$ implies that all the terms are zero. It is clear that this condition is necessary for observations arising from utility maximization. What is less clear, and indeed surprising, is that it is also sufficient.

Theorem 2 (Afriat's Theorem) *If the data set D satisfies the Generalized Axiom then there exists a piecewise linear, continuous, strictly monotone and concave utility function that generates the observations.*

This is a remarkable result because it gives succinct, testable conditions that a finite data set must satisfy in order to be consistent with utility maximization. Moreover, from the result, it follows that the assumptions of continuity, monotonicity and concavity are not refutable by a finite data set.

Afriat's original argument begins by asserting the existence of numbers ϕ_1, $\dots$, ϕ_n, and $\lambda_1, \dots, \lambda_n > 0$ that satisfy the following unusual system of linear inequalities (from now Afriat inequalities)

$$\phi_j \leq \phi_i + \lambda_i a_{ij}, \text{ for all } i, j \in N .$$

He then defines the utility function

$$u(x) = \min\{\phi_1 + \lambda_1 p_1 \cdot (x - x_1), \dots, \phi_n + \lambda_n p_n \cdot (x - x_n)\} .$$

We notice that each term in this expression is linear (and hence continuous and concave) and strictly monotone. Therefore, u, as their pointwise minimum, is continuous, concave, and strictly monotone as well. Finally, as is shown in the next two steps, u indeed generates the observations in the data set D.

1. $u(x_j) = \phi_j$, for all $j \in N$.

   By definition $u(x_j) = \min_i\{\phi_i + \lambda_i p_i \cdot (x_j - x_i)\} = \phi_j + \lambda_j p_j \cdot (x_j - x_j) = \phi_j$, where the minimum is taken by the index j from the Afriat inequalities.
2. $p_j \cdot x \leq p_j \cdot x_j \Rightarrow u(x) \leq u(x_j)$.

   $u(x) \leq \phi_j + \lambda_j p_j \cdot (x - x_j) \leq \phi_j = u(x_j)$, where the first inequality follows from the definition of u, the second from the fact that x is feasible at prices p_j and the last equality from Step 1.

2 A simple case

We have shown that the Afriat inequalities imply the existence of a nice utility function that generates the data. What is less straightforward is to show that if the observations satisfy the Generalized Axiom then the Afriat inequalities have a solution. Afriat's original proof is an inductive one, which is correct in the case in which $a_{ij} \neq 0$, $i \neq j$. Indeed in this case the proof is quite simple.[2]

[2] A similar version was presented in an informal communication by M. Weitzman.

Claim 1. There is an index $i \in N$ with $a_{ij} \geq 0$ for all $j \in N$.

Proof of Claim 1. If this were not so, then every row would have a strictly negative entry. Start with row i, say, and suppose that $a_{ij} < 0$. Now consider row j, and identify a negative entry, say $a_{jk} < 0$. Continue to generate the sequence $i, j, k, ...,$ until an index is repeated. Then a subsequence of this sequence yields a contradiction to the Generalized Axiom. $\square$

The existence of λ_j and ϕ_j is trivially true for $n = 1$; we can choose $\lambda_1 = 1$ and ϕ_1 arbitrarily. For the induction let us begin by renumbering the observations (and hence the rows and columns of A) so that $a_{nj} > 0$ for $j = 1, ..., n-1$ (using Claim 1). Now suppose, by induction, that there exist $\phi_1, ..., \phi_{n-1}; \lambda_1, ..., \lambda_{n-1} > 0$ such that

$$\phi_j \leq \phi_i + \lambda_i a_{ij},\ i \neq j,\ i, j = 1, ..., n-1.$$

Let us select ϕ_n such that

$$\phi_n \leq \min_{i=1,...,n-1} \phi_i + \lambda_i a_{in},$$

and then choose $\lambda_n > 0$ so that

$$\phi_j \leq \phi_n + \lambda_n a_{nj},\ \text{for } j = 1, ..., n-1.$$

Since all the non-diagonal elements of the nth row are strictly positive, λ_n can be chosen large enough so that these $n-1$ inequalities hold. Note the difficulty that arises if any a_{nj} is zero: increasing λ_n will not help to fix the inequality for this n and j. This completes the proof that the Afriat inequalities have a solution in this simple case.

The general case, in which non-diagonal elements are allowed to be zero, is related to the issue of indifference classes in the revealed preference ordering. Two authors, Varian [5] and Diewert [2], have given correct proofs in this general case. They prove the result using an inductive argument which manages to handle the subtle issue of indifference classes. Unfortunately, the induction in each of these presentations is complex and may involve the introduction of more than one price-quantity observation at each step.

3 A general inductive proof

We now provide a simple proof for Afriat's theorem in the general case where $a_{nj} \geq 0$ for $j = 1, ..., n-1$, but with some of these entries possibly zero. The argument is inductive, and as in the simple case, the inductive step introduces a single observation at a time.

The key is to apply the inductive hypothesis to a different $(n-1) \times (n-1)$ matrix A'. Specifically, for $j = 1, ..., n-1$, we define

$$a'_{ij} := \begin{cases} a_{ij} & \text{if } a_{nj} > 0, \\ \min\{a_{ij}, a_{in}\} & \text{if } a_{nj} = 0. \end{cases} \tag{1}$$

Claim 2. A' satisfies the Generalized Axiom.

Proof of Claim 2. First note that, if $a_{nj} = 0$, then $a_{jn} \geq 0$ by the Generalized Axiom, so that $a'_{jj} = a_{jj} = 0$ for $j = 1, \ldots, n-1$. Now suppose that A' has a cycle $(i, j, k, \ldots, r, i)$ with

$$\begin{aligned} a'_{ij} &\leq 0 \\ a'_{jk} &\leq 0 \\ &\vdots \\ a'_{ri} &\leq 0 \end{aligned}$$

and at least one term strictly negative. Since A does satisfy the Generalized Axiom by hypothesis, there must be a term, say that for (p, q), with

$$a'_{pq} \neq a_{pq} \ .$$

But if $a'_{pq} = a_{pn}$ and $a_{nq} = 0$, then we can replace the cycle $(\ldots, p, q, \ldots)$ by $(\ldots, p, n, q, \ldots)$ with two new terms

$$\begin{aligned} a_{pn} &\leq 0 \\ a_{nq} &= 0 \end{aligned}$$

and, as before, at least one of the terms in the new sequence is strictly negative. Continuing in this way we can construct a cycle in A that violates the Generalized Axiom, contrary to our assumption. Hence A' must satisfy the Generalized Axiom. $\square$

We can therefore apply our inductive assumption to A' to guarantee the existence of ϕ_i and positive λ_i for $i \in N_- := \{1, 2, ..., n-1\}$ so that

$$\phi_j \leq \phi_i + \lambda_i a'_{ij} \tag{2}$$

for $i, j \in N_-$. Since $a'_{ij} \leq a_{ij}$ from (1), this ensures that the Afriat inequalities hold also for A for $i, j \in N_-$. Next, set

$$\phi_n = \min_{i \in N_-} \{\phi_i + \lambda_i a_{in}\}$$

(note that we choose equality, not less than or equal to), to achieve the inequalities for $i < n$, $j = n$. Finally, set

$$\lambda_n := \max \left\{ 1, \max_{j \in N_-, a_{nj} > 0} [(\phi_j - \phi_n)/a_{nj}] \right\} .$$

As in the simple case, this choice makes sure that the inequalities hold for $i = n$ and $j < n$ in the case that $a_{nj} > 0$. To complete the proof, suppose that $a_{nj} = 0$. Then we have

$$\begin{aligned} \phi_j &\leq \min\nolimits_{i \in N_-} \{\phi_i + \lambda_i a'_{ij}\} && \text{(by (2))} \\ &\leq \min\nolimits_{i \in N_-} \{\phi_i + \lambda_i a_{in}\} && \text{(by (1))} \\ &= \phi_n && \text{by definition of } \phi_n \\ &= \phi_n + \lambda_n a_{nj} && \text{since } a_{nj} = 0. \end{aligned}$$

Clearly the inequality holds for $i = j = n$, and so the inductive step is complete. This finishes the proof.

4 A proof using linear programming

Diewert's proof [2] relates the Afriat inequalities to a particular linear programming problem. However the programming problem is not directly used in his proof. The argument presented here makes use of a linear program which is essentially identical to Diewert's, but uses the Duality Theorem of Linear Programming to show that the Afriat inequalities have a solution.[3]

Consider the following linear programming problem:

$$\begin{array}{rll} \min_{\lambda,\phi} & 0\cdot\lambda + 0\cdot\phi & \\ & \lambda_i & \geq 1, \text{ for all } i \in N, \\ a_{ij}\lambda_i + \phi_i - \phi_j & & \geq 0, \text{ for all } i, j \in N \text{ with } i \neq j \end{array}$$

in which the objective function is zero and the constraints are the Afriat inequalities. We shall show that the dual linear program is feasible and has a maximum of zero. The Duality Theorem then implies that the original problem is also feasible, and therefore the Afriat inequalities have a solution. Although the argument may seem a bit eccentric, the procedure is a standard trick to verify that a system of linear inequalities is consistent.

The matrix associated with the linear program is

$$\begin{array}{l} \text{objective} \\ \\ \\ \vdots \\ \\ \\ \vdots \\ \\ \vdots \\ \\ \vdots \\ \\ \text{variables} \end{array}
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & RHS \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
a_{12} & 0 & \cdots & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{1n} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n1} & -1 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n,n-1} & 0 & 0 & \cdots & -1 & 1 & 0 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n & \phi_1 & \phi_2 & \cdots & \phi_{n-1} & \phi_n &
\end{bmatrix}
\begin{array}{c} \\ y_1 \\ y_2 \\ \vdots \\ y_n \\ x_{12} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{n,n-1} \\ \\ \end{array}$$

In this matrix the top row describes the coefficients of the objective function, the bottom row the variables associated with the columns and the last column the right hand side of the inequalities. The slack variables have been omitted.

[3] Our colleague, John Geanakoplos, has shown us an elegant proof that the Afriat inequalities have a solution using the Min-Max Theorem for two-person zero-sum games.

If the dual variable associated with the inequality $\lambda_i \geq 1$ is $y_i (\geq 0)$ and the dual variable associated with the inequality $a_{ij}\lambda_i + \phi_i - \phi_j \geq 0$, for $i \neq j$, is $x_{ij} (\geq 0)$, the dual problem can be stated as

$$
\begin{array}{lrcl}
\max_{y,x} \sum_{i \in N} y_i & & & \\
 & \sum_{h \in N} x_{hi} - & \sum_{j \in N} x_{ij} & = 0, \text{ for all } i \in N, \\
y_i & & + \sum_{j \in N} a_{ij} x_{ij} & = 0, \text{ for all } i \in N, \\
\text{with } y_i, x_{ij} \geq 0 \text{ for all } i, j. & & &
\end{array}
$$

The dual variables x_{ij} can be viewed as the entries in an $n \times n$ matrix X, whose diagonal entries are zero and whose off-diagonal elements are non-negative. The first set of constraints in the dual problem state that for each i the sum of the entries in row i of X equals the sum of the entries in column i.

In order to use the Duality Theorem to prove that the Afriat inequalities have a solution, we need to show that $x = 0$, $y = 0$ is the optimal solution to the dual problem. Clearly $x = 0$, $y = 0$ is feasible for the dual and 0 is an lower bound for the optimal value of the dual objective function.

Claim 3. Let (x, y) be a feasible solution to the dual linear program. Then there is a feasible solution, possibly different, with the same objective function value and with no cycle $(i, j), (j, k), \ldots, (r, i)$ on which all x_{pq}'s are positive and all a_{pq}'s zero.

Proof of Claim 3. If there is such a cycle in a feasible solution, we can decrease each x_{pq} on the cycle by the minimum value of these x_{pq}'s, so that at least one such value becomes zero. In this procedure, the perturbed matrix X will still satisfy the constraints of the dual problem and the variables y_p, and hence the objective function value, are unchanged since we are only modifying those x_{pq}'s whose corresponding a_{pq} coefficient is zero. □

Now let us show that an optimal solution to the dual problem is $x = 0$, $y = 0$. Suppose, to the contrary, that $y_i > 0$ in some feasible solution (x, y), which without loss of generality we can assume satisfies the property of Claim 3. Then the sum

$$\sum_{q \in N} a_{iq} x_{iq} < 0$$

and at least one term is negative, say $a_{ij} x_{ij}$. Therefore a_{ij} is negative and x_{ij} positive. By the first set of constraints,

$$\sum_{q \in N} x_{jq} > 0,$$

while

$$\sum_{q \in N : x_{jq} > 0} a_{jq} x_{jq} \leq 0$$

by the second set of constraints. We can therefore choose $k \neq j$ with x_{jk} positive and a_{jk} nonpositive. Continuing in this way, we must eventually repeat an index, and therefore we construct a cycle $(\ell, m, \ldots, r, \ell)$ on which all x_{pq}'s are positive and all a_{pq}'s nonpositive.

If the index we repeat is the first one with which we started, we immediately get a contradiction since the Generalized Axiom implies that all the terms in the cycle must be zero, but the first one is strictly negative by construction.

In the case that the cycle we construct does not include the first term, again, the Generalized Axiom implies that all terms must be zero, but this was already ruled out by our assumption that (x, y) satisfies the property of Claim 3.

We have demonstrated that the dual linear program is feasible and its maximum value is 0. By the Duality Theorem of Linear Programming the original problem is feasible, which means that the Afriat inequalities have a solution.

5 Complexity

Here we discuss the complexity of determining whether the data D is consistent with utility maximization and, if so, computing a possible utility function u.

We remarked in the introduction that the Generalized Axiom gives testable conditions for the data D to be consistent with utility maximization. But how hard is it to check whether the axiom holds, and if so, to find a possible utility function? At first sight, we need to check every possible cycle, and while this is a finite procedure, there are exponentially many cycles. If we knew the $2n$ numbers $\phi_1, \ldots, \phi_n$ and $\lambda_1, \ldots, \lambda_n > 0$, potentially satisfying the Afriat inequalities, then we would merely have to check these n^2 relations, and from these a suitable utility function is at hand. Diewert [2] proposed to find these numbers by solving a linear programming problem, but this is computationally burdensome. Varian's proof [5] gives an $O(n^3)$ algorithm to find the ϕ's and λ's. Indeed, Varian first defines x_i to be directly revealed preferred to x_j if $p_i \cdot x_j \leq p_i \cdot x_i$, and then computes the transitive closure R of this relation by a graph-theoretic algorithm in $O(n^3)$ time. Then the Generalized Axiom can be checked simply: for each i and j, see if $x_i R x_j$ and $p_j \cdot x_i < p_j \cdot x_j$; if so the Generalized Axiom is violated. If this does not occur for any such pair, the Generalized Axiom is satisfied. Armed with the transitive closure, Varian finds the ϕ's and λ's by an algorithm that must consider together every subset of observations with each pair related by R. Our inductive proof in Section 3 provides a simple alternative $O(n^3)$ method that determines these parameters one by one. (Of course, we also need $O(n^2)$ work to compute the entries of A from the data D.)

At each step of the inductive process, we search the current matrix A to find a nonnegative row, say the ith, which takes $O(n^2)$ time. (If there is no such row, then we can find a cycle violating the Generalized Axiom by the argument in the proof of Claim 1, also in $O(n^2)$ time.) We then interchange the ith and nth rows of A, in $O(n)$ time, and calculate the reduced matrix A', in $O(n^2)$ time. When we receive information back from the smaller problem, we can find ϕ_n and λ_n each in $O(n)$ time. (If the smaller problem returns a cycle violating the Generalized Axiom in A', we can expand this to a cycle violating the Generalized Axiom in A using the argument in the proof of Claim 2, also in $O(n)$ time.) This gives a total amount of work at each stage of $O(n^2)$, for a total complexity of $O(n^3)$.

However, if at each stage we can find a positive row (except for its diagonal entry), then we can avoid the per stage $O(n^2)$ work and complete all the computation in a total of $O(n^2)$ time. Clearly we do not require the $O(n^2)$ work to calculate A' so we only need to show how the search for a positive row can be performed in only $O(n)$ time at each stage. Initially, let us compute the number of negative and zero entries in each row, at a one-time cost of $O(n^2)$. Then at each stage we can scan these counts to find a positive row, and then after permuting that row and the associated column to the end, we can update the counts for the submatrix containing all but the last row and column in just $O(n)$ work. Hence there is only $O(n)$ work per stage for a total of $O(n^2)$. (This complexity also holds if there are only a fixed number of times that a positive row cannot be found.)

When can we use this simplified algorithm? Clearly, if A contains no zero elements outside its diagonal, then the Generalized Axiom implies the existence of a positive row. More generally, note that, if the Generalized Axiom holds vacuously, i.e., there are no cycles with all a_{ij}'s nonpositive at all, then the argument of the proof of Claim 1 shows that a positive row exists. This condition (assuming that all demand vectors x_i are distinct) is usually called the Strong Axiom of Revealed Preference (see, e.g., Varian [5]). Thus either the simple case considered in Section 3 or the Strong Axiom leads to the reduced complexity of $O(n^2)$ time to compute the ϕ's and λ's satisfying the Afriat inequalities and hence a possible utility function.

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