THE ECONOMY

- N consumers
- Two goods. Good A is public, good x is private
- Preferences are described by $u_i = \log x_i + \theta \log(A), \theta > 0$
- Each consumer has $\overline{x} > 0$ units of the private good
- The public good is produced out of the private good with technology described by the production function $\hat{A} = \frac{\hat{x}}{k}, k > 0$

EFFICIENT POINTS

$$\max \sum_{i=1}^{N} \alpha_i u_i = \sum_{i=1}^{N} \alpha_i \log x_i + \theta \log(A), \text{subject to } \hat{x} + \sum_{i=1}^{N} x_i \le N\overline{x}, A \le \hat{A} = \frac{\hat{x}}{k}, \text{ie}$$

$$\boxed{\begin{array}{l} \text{pareto efficient points} \\ \sum_{i=1}^{N} x_i^p = \frac{N\overline{x}}{1+\theta}, A^p = \frac{\theta N \overline{x}}{k(1+\theta)} \end{array}} \qquad (1)$$

$$\boxed{\begin{array}{l} \text{symmetric pareto efficient point} \\ x_i^p = \frac{\overline{x}}{1+\theta}, A^p = \frac{\theta N \overline{x}}{k(1+\theta)} \\ u_i^p = \log\left(\frac{\overline{x}}{1+\theta}\right) + \theta \log\left(\frac{\theta N \overline{x}}{k(1+\theta)}\right) \end{array}} \qquad (2)$$

COMPETITIVE EQUILIBRIUM

Let $0 \le s < k$ be a subsidy paid to the firm per unit of output. The subsidy is financed by a lump-sum tax $0 \le T < \overline{x}$ on each consumer, equal for all consumers. We will compute competitive equilibria under such a tax-subsidy scheme

1. NAME THE PRICE OF EACH GOOD

$p = \text{price of } A, w = \text{price of } x$

2. NORMALIZE PRICES (OPTIONAL)

$w = 1$

3. SOLVE THE OPTIMIZATION PROBLEMS OF FIRMS

$\Pi = p\hat{A} - w\hat{x} + s\hat{A} = (p + s - k)\hat{A}$ *is maximized at*

$$\hat{A}=\begin{cases}0 & \text{if} \quad p<k-s\\ \geq 0 & \text{if} \quad p=k-s\\ \infty & \text{if} \quad p>k-s\end{cases} \qquad \hat{x}=k\hat{A} \tag{3}$$

4. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

max $U_i=\log x_i+\theta\log A$, subject to $x_i+pA_i\leq\bar{x}-T, A=A_i+\sum_{j\neq i}^{N}A_j$ The solution is

$$(x_i,A_i)=\begin{cases}\left(\dfrac{\bar{x}-T+p\sum_{j\neq i}^{N}A_j}{1+\theta},\dfrac{\theta(\bar{x}-T)-p\sum_{j\neq i}^{N}A_j}{p(1+\theta)}\right) & \text{if} \quad \sum_{j\neq i}^{N}A_j<\dfrac{\theta(\bar{x}-T)}{p}\\ (\bar{x}-T,0) & \text{if} \quad \sum_{j\neq i}^{N}A_j\geq\dfrac{\theta(\bar{x}-T)}{p}\end{cases} \tag{4}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\sum_{i=1}^{N}x_i+\hat{x}=N\bar{x}, \sum_{i=1}^{N}A_i=\hat{A}=\frac{\hat{x}}{k} \tag{5}$$

There is a unique solution, described by

$$\boxed{\begin{array}{l}\text{competitive equilibrium at subsidy level } s\\ p=k-s, A^e=\dfrac{\theta N\bar{x}}{N(k-s)+\theta k}\\ A^e{}_i=\dfrac{\theta\bar{x}}{N(k-s)+\theta k}, x^e{}_i=\dfrac{N(k-s)\bar{x}}{N(k-s)+\theta k}\\ u^e{}_i=\log(\dfrac{N(k-s)\bar{x}}{N(k-s)+\theta k})+\theta\log(\dfrac{\theta N\bar{x}}{N(k-s)+\theta k})\end{array}} \tag{6}$$

$$\boxed{\begin{array}{l}\text{competitive equilibrium at subsidy level } s=0\\ p=k, A^e=\dfrac{\theta N\bar{x}}{(N+\theta)k}\\ A^e{}_i=\dfrac{\theta\bar{x}}{(N+\theta)k}, x^e{}_i=\dfrac{N\bar{x}}{(N+\theta)}\\ u^e{}_i=\log(\dfrac{N\bar{x}}{(N+\theta)})+\theta\log(\dfrac{\theta N\bar{x}}{(N+\theta)k})\end{array}} \tag{7}$$

To see that this is in fact the only equilibrium, note that

●there is no equilibrium such that $p > k - s$, because

$$\sum_{i=1}^{N} A_i < \infty = \hat{A}$$

●There is no equilibrium such that $p < k - s$, because

$\sum_{i=1}^{N} A_i = \hat{A} = 0$ implies $A_i = 0 \ \forall i$,violating condition (4) ,namely $\sum_{j \neq i}^{N} A_j \geq \frac{\theta(\overline{x} - T)}{p}$

●The only remaining possibility is $p = k - s$ .Let n denote the number of consumers that contribute to the public good, i.e.

$$\begin{aligned} A_i &> 0, i = 1,...,n \\ A_i &= 0, i = n+1,...,N \end{aligned} \tag{8}$$

We will show that

(1) All positive A_i are equal, and (2) All A_i are positive,i.e. n=N

For all $i = 1,...,n$ we have, by (4),

$$A_i = \frac{\theta(\overline{x} - T) - p\sum_{j \neq i}^{N} A_j}{p(1+\theta)} = \frac{\theta(\overline{x} - T)}{p(1+\theta)} - \frac{A - A_i}{(1+\theta)}$$

Solving for A_i we obtain

$$A_i = \frac{(\overline{x} - T)}{p} - \frac{A}{\theta}, i = 1,...,n \tag{9}$$

Hence all positive A_i are equal, and $A = nA_i$ .By (9),then,

$$A_i = \frac{(\overline{x} - T)}{p} \frac{\theta}{n+\theta}, i = 1,...,n \tag{10}$$

Suppose now,for contradiction, that n<N.By(8),(10) we have

$$\begin{aligned} \sum_{j \neq i}^{N} A_j &= \frac{(\overline{x} - T)}{p} \frac{n\theta}{n+\theta} - A_i, i = 1,...,k \\ \sum_{j \neq i}^{N} A_j &= \frac{(\overline{x} - T)}{p} \frac{n\theta}{n+\theta}, i = n+1,...,N \end{aligned} \tag{11}$$

By(4),(8)

$$\sum_{j\neq i}^{N} A_j < \frac{\theta(\overline{x}-T)}{p}, i=1,...,n$$
$$\sum_{j\neq i}^{N} A_j \geq \frac{\theta(\overline{x}-T)}{p}, i=n+1,...,N \tag{12}$$

By (11),(12) and the facr that p=k-s

$$\frac{(\overline{x}-T)}{k-s}\frac{n\theta}{n+\theta} - A_i < \frac{\theta(\overline{x}-T)}{k-s}, i=1,...,n \tag{13}$$

$$\frac{(\overline{x}-T)}{k-s}\frac{n\theta}{n+\theta} \geq \frac{\theta(\overline{x}-T)}{k-s}, i=n+1,...,N \tag{14}$$

Inequality (14) implies the contradiction $\frac{n}{n+\theta} \geq 1$,hence

$$n = N \tag{15}$$

By (15),(10) we obtain

$$A_i = \frac{(\overline{x}-T)}{k-s}\frac{\theta}{N+\theta}, i=1,...,N \tag{16}$$

The lump-sum tax T must satisfy

$$NT = sNA_i \tag{17}$$

By (16),(17) we obtain

$$A_i = \frac{\theta\overline{x}}{N(k-s)+\theta k} \tag{18}$$

$$T = \frac{s\theta\overline{x}}{N(k-s)+\theta k} \tag{19}$$

We must now check that (13) ,(18),(19) hold together, i.e. that the necessary and sufficient condition (13) for positive demand for the public good is satisfied by the unique candidate solution (18),(19).A bit of algebra shows that (13) ,(18),(19) are equivalent to the condition $\theta^2 + \theta N + k + \theta > 0$,which is obviously true. Hence there exists a unique equilibrium, given by(6).Comparing (1) *with* (6) we can deduce the efficiency properties of private provision of public goods.

PRIVATE PROVISION OF PUBLIC GOODS AND WELFARE

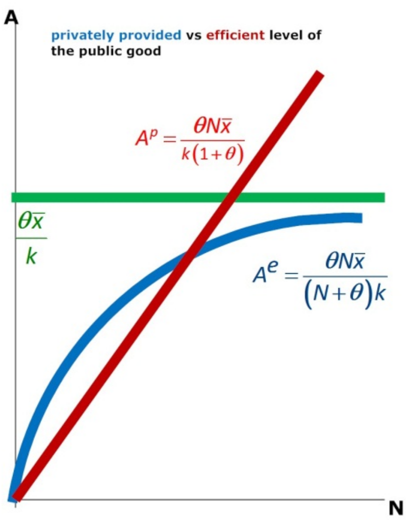
1.the uncorrected competitive equilibrium(s = 0) is inefficient.

Measured by the size of underprovision of the public good, welfare loss is

$$\text{welfare loss}=A^p - A^e = \frac{\theta N\bar{x}(N-1)}{k(1+\theta)(N+\theta)} \to \infty \text{ as } N \to \infty$$
$$\text{percentage welfare loss} = \frac{A^p - A^e}{A^e} = \frac{(N-1)}{(1+\theta)} \to \infty \text{ as } N \to \infty \qquad (20)$$

Measured in utility terms,welfare loss can be defined by comparing competitive equilibrium utilities at s=0 ,as given by (7) ,to utilities at the symmetric pareto point,as given by (2)

$$\text{welfare loss}=u_i^p - u_i^e = \log\left(\frac{N+\theta}{N(1+\theta)}\right) + \theta\log\left(\frac{N+\theta}{1+\theta}\right) \to \infty \text{ as } N \to \infty$$
$$\text{percentage welfare loss}=\frac{u_i^p - u_i^e}{u_i^e} = \frac{\log\left(\frac{\bar{x}}{1+\theta}\right) + \theta\log\left(\frac{\theta N\bar{x}}{k(1+\theta)}\right)}{\log\left(\frac{N\bar{x}}{(N+\theta)}\right) + \theta\log\left(\frac{\theta N\bar{x}}{k(N+\theta)}\right)} - 1 \to \infty \text{ as } N \to \infty$$



2.correction of competitive equilibrium

We can compute the level of the subsidy s that restores the efficiency of competitive equilibrium by comparing (1) to (6) and solving for s the equation $A^e = A^p$,i.e. $\frac{\theta N\bar{x}}{k(1+\theta)} = \frac{\theta N\bar{x}}{N(k-s)+\theta k}$

$$s = k\left(1 - \frac{1}{N}\right) \qquad (21)$$

The corresponding equilibrium is, by (6) *and* (21),

$$\boxed{\begin{array}{l}\text{corrected competitive equilibrium}\\ s=k\left(1-\frac{1}{N}\right), p=\frac{k}{N}, A^{e}=\frac{\theta N\overline{x}}{k(1+\theta)}\\ A_{i}^{e}=\frac{\theta\overline{x}}{k(1+\theta)}, x_{i}^{e}=\frac{\overline{x}}{1+\theta}, T=\left(1-\frac{1}{N}\right)\frac{\theta\overline{x}}{1+\theta}\end{array}} \qquad (22)$$

3. optimal size of the public sector

Define the optimal size of the public sector to be the ratio

$$\boxed{\frac{\text{tax revenue}}{\text{value of total endowment}}=\frac{NT}{N\overline{x}}=\frac{T}{\overline{x}}=\left(1-\frac{1}{N}\right)\frac{\theta}{1+\theta}} \qquad (23)$$

It increases to $\frac{\theta}{1+\theta}$ *as* $N\to\infty$*.Hence in large economies the optimal size of the public sector can be any number between 0 and 1,depending on the value of the parameter* θ