## COMPETITIVE EQUILIBRIUM WITH EXTERNALITIES

### THE ECONOMY

- two goods, A, X
- One consumer
- Two firms,1 and 2

Firm 1 produces good A out of good X with production function

$$A_1 = X_1, X_1 \geq 0 \quad (1)$$

where $X_1$ is the quantity of good X used as input in the production of good A, and $A_1$ is the quantity produced of good A.

Firm 2 produces good A out of good X with technology described by the production function

$$A_2 = \frac{2X_2}{1+X_1}, X_2 \geq 0 \quad (2)$$

where $X_2$ is the quantity of good X used as input in the production of good A, and $A_2$ is the quantity produced of good A.

negative production externality: the production function of firm 2,as defined in (2), is a strictly decreasing function of a variable, namely $X_1$, chosen by firm 1.

### Consumer

- Consumption set $R_+^2$
- Endowment vector $\omega = \overbrace{[0, \bar{X}]}^{A,X}, \bar{X} > 0$
- Ownership shares: $\theta_1 = 1 = \theta_2$
- Utility function $u(A, X, A_1, X_1, A_2, X_2) = AX$

where $X$ is the quantity consumed of good X, and $A$ is the quantity consumed of good A.

# COMPETITIVE EQUILIBRIUM

## 1.NAME THE PRICE OF EACH GOOD

$p =$ price of commodity A,$w$=price of commodity X.Normalize $p = 1$

## 2. DEFINE CONSUMER INCOME

$$M = w\overline{X} + \Pi_1 + \Pi_2 \tag{3}$$

## 3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

max $u = \log(A) + \log(X)$,subject to $A + wX \leq M, A \geq 0, X \geq 0$

Variables: $A, X$

parameters: $w, M$

Conditions on parameters: $w > 0, M > 0$

The solutions are

$$(A, X) = \left(\frac{M}{2}, \frac{M}{2w}\right) \tag{4}$$

## 4.1. SOLVE THE OPTIMIZATION PROBLEM OF FIRM 1

max $\Pi_1 = A_1 - wX_1$

subject to $X_1 \geq 0, A_1 \geq 0, A_1 = X_1$

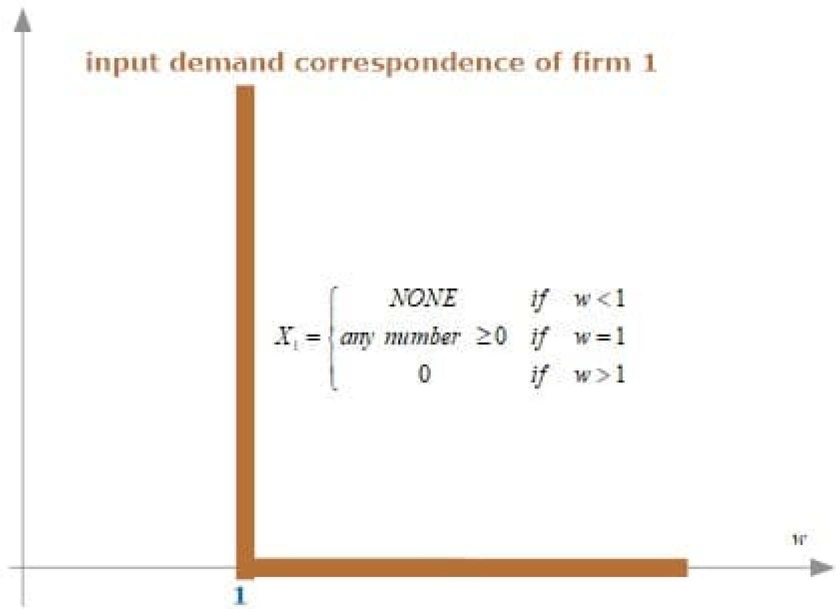
Variables:$A_1, X_1$

parameters: $w$

conditions on parameters:$w > 0$

The solution is

$$(A_1, X_1, \Pi_1) = \begin{cases} \text{NONE} & \text{if} \quad w < 1 \\ (X_1, X_1, 0) & \text{if} \quad w = 1 \\ (0, 0, 0) & \text{if} \quad w > 1 \end{cases} \tag{5}$$




## 4.2. SOLVE THE OPTIMIZATION PROBLEM OF FIRM 2

$\max \Pi_2 = A_2 - wX_2$

subject to $A_2 \geq 0, X_2 \geq 0, A_2 = \dfrac{2X_2}{1+X_1}$

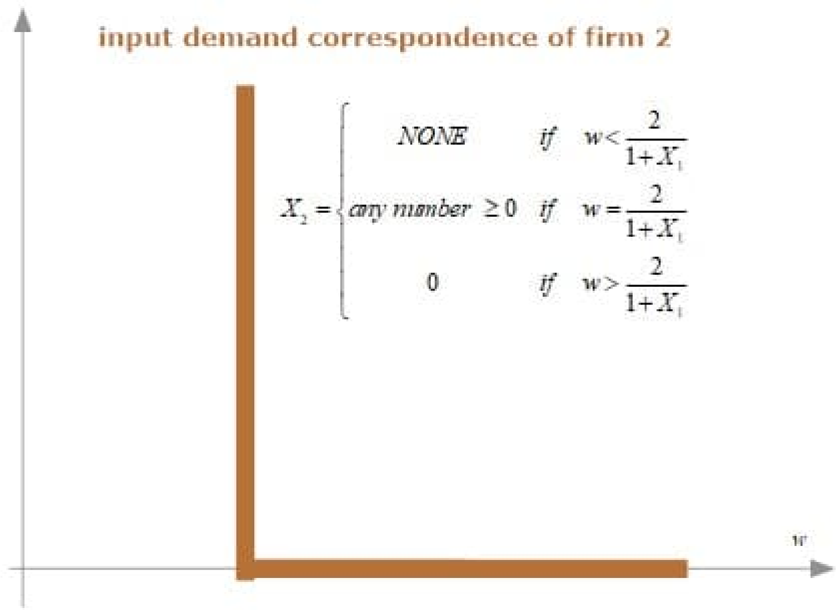
Variables: $A_2, X_2$

parameters: $w, X_1$

conditions on parameters: $w > 0, X_1 \geq 0$

The solution is

$$(A_2, X_2, \Pi_2) = \begin{cases} \text{NONE} & \text{if} \quad w < \dfrac{2}{1+X_1} \\ \left(\dfrac{2X_2}{1+X_1}, X_2, 0\right) & \text{if} \quad w = \dfrac{2}{1+X_1} \\ (0,0,0) & \text{if} \quad w > \dfrac{2}{1+X_1} \end{cases} \tag{6}$$




## 5. SOLVE THE EQUILIBRIUM CONDITIONS

$$A = A_1 + A_2 \tag{7}$$

$$X + X_1 + X_2 = \overline{X} \tag{8}$$

Search for solutions of the equilibrium equations.

By (6),(5)

all equilibria,if they exist,must satisfy

$$\boxed{w \geq 1 \text{ and } w \geq \frac{2}{1+X_1}, \Pi_1 = 0 = \Pi_2, M = w\overline{X}} \tag{9}$$

Hence, we can restrict the search to the space (9).

Hypothesis 1 (to be accepted or rejected, depending on the outcome of the search):

there is an equilibrium with $\frac{2}{1+X_1} < 1$ ,i.e. $X_1 > 1$

Solution 1

Given the hypothesis, the possibilities for w are


By (9),there are no equilibria when $w<1$

By (6),(5),there is no equilibrium with $w>1$ ,because then

$$(A_1,X_1,\Pi_1)=(0,0,0)=(A_2,X_2,\Pi_2),A=\frac{w\overline{X}}{2}$$

if $w=1$ then $(A_2,X_2,\Pi_2)=(0,0,0),(A_1,X_1,\Pi_1)=(X_1,X_1,0),A=X=\frac{\overline{X}}{2}$ .Then (8) becomes

$X_1=\frac{\overline{X}}{2}$ ,and this is the only candidate equilibrium.

Consistency check 1

$\frac{2}{1+X_1}<1$ becomes, at the only candidate equilibrium, $\frac{2}{1+\frac{\overline{X}}{2}}<1$ ,i.e., $\overline{X}>2$

Conclusion 1: we accept hypothesis 1 iff $\overline{X}>2$.The corresponding equilibrium is

$$\boxed{\begin{array}{l} \text{type 1 competitive equilibrium,it exists iff } \boxed{\overline{X}>2} \\ w=1 \\ (A_2,X_2,\Pi_2)=(0,0,0) \\ (A_1,X_1,\Pi_1)=(\frac{\overline{X}}{2},\frac{\overline{X}}{2},0) \\ (A,X)=\left(\frac{\overline{X}}{2},\frac{\overline{X}}{2}\right) \\ U_{EQ1}=\frac{(\overline{X})^2}{4} \end{array}} \qquad (10)$$

We continue the search for equilibria with a new hypothesis

Hypothesis 2 (to be accepted or rejected, depending on the outcome of the search):

there is an equilibrium with $\frac{2}{1+X_1}>1$

## Solution 2

Given the hypothesis, the possibilities for w are


By (9),there are no equilibria when $w < \frac{2}{1+X_1}$.

By (6),(5),there is no equilibrium with $w > \frac{2}{1+X_1}$ ,because then $(A_1, X_1, \Pi_1) = (0,0,0) = (A_2, X_2, \Pi_2), A = \frac{w\bar{X}}{2}$ .

if $w = \frac{2}{1+X_1}$ then $(A_2, X_2, \Pi_2) = (2X_2, X_2, 0), (A_1, X_1, \Pi_1) = (0,0,0), w = 2, A = \bar{X}, X = \frac{\bar{X}}{2}$

Then (8) becomes $X_2 = \frac{\bar{X}}{2}$ ,and this is the only candidate equilibrium.

## Consistency check 2

$\frac{2}{1+X_1} > 1$ becomes, at the only candidate equilibrium, $\frac{2}{1+0} > 1$ ,which is always true.

Conclusion 2: we accept hypothesis 2 without qualifications. The corresponding equilibrium is

$$\begin{array}{|l|}\hline \text{type 2 competitive equilibrium,it exists iff } \bar{X} > 0 \\ \hline w = 2 \\ (A_2, X_2, \Pi_2) = (\bar{X}, \frac{\bar{X}}{2}, 0) \\ (A_1, X_1, \Pi_1) = (0,0,0) \\ (A, X) = \left(\bar{X}, \frac{\bar{X}}{2}\right) \\ U_{EQ2} = \frac{(\bar{X})^2}{2} \\ \hline \end{array} \qquad (11)$$

We continue the search for equilibria with a new hypothesis

Hypothesis 3 (to be accepted or rejected, depending on the outcome of the search):

there is an equilibrium with $\frac{2}{1+X_1}=1$

Solution 3

Given the hypothesis, $X_1=1$,and the possibilities for w are


By (9),there are no equilibria when $w<1$.

By (6),(5),there is no equilibrium with $w>1$ ,because then

$\left(A_1,X_1,\Pi_1\right)=\left(0,0,0\right)=\left(A_2,X_2,\Pi_2\right),A=\frac{w\bar{X}}{2}$

if $w=1$ then $\left(A_2,X_2,\Pi_2\right)=\left(X_2,X_2,0\right),\left(A_1,X_1,\Pi_1\right)=\left(1,1,0\right),A=X=\frac{\bar{X}}{2}$ .Then (8) becomes $1+X_2=\frac{\bar{X}}{2}$ ,and therefore $X_2=\frac{\bar{X}-2}{2}$ and this is the only candidate equilibrium.

Consistency check 3

$X_2=\frac{\bar{X}-2}{2}\geq 0$ implies $\bar{X}\geq 2$

Conclusion 3: we accept hypothesis 3 iff $\bar{X}\geq 2$ The corresponding equilibrium is

$$
\boxed{
\begin{array}{l}
\underline{\text{type 3 competitive equilibrium,it exists iff } \boxed{\bar{X} \geq 2}} \\
w = 1 \\
(A_2, X_2, \Pi_2) = (\frac{\bar{X}-2}{2}, \frac{\bar{X}-2}{2}, 0) \\
(A_1, X_1, \Pi_1) = (1,1,0) \\
(A, X) = \left( \frac{\bar{X}}{2}, \frac{\bar{X}}{2} \right) \\
U_{EQ3} = \frac{(\bar{X})^2}{4}
\end{array}
} \qquad (12)
$$

We end the search for equilibria here since we have exhausted the search space. Comparing (12),(11) and (10) we conclude that ,as far as welfare is concerned, there exist in fact only two types of equilibria

$$
\boxed{
\begin{array}{l}
\underline{\text{low wage competitive equilibrium,it exists iff } \boxed{\bar{X} \geq 2}} \\
w = 1 \\
\boxed{
\begin{array}{l}
(A_2, X_2, \Pi_2) = (0,0,0), (A_1, X_1, \Pi_1) = (\frac{\bar{X}}{2}, \frac{\bar{X}}{2}, 0) \\
\text{OR} \\
(A_2, X_2, \Pi_2) = (\frac{\bar{X}-2}{2}, \frac{\bar{X}-2}{2}, 0), (A_1, X_1, \Pi_1) = (1,1,0)
\end{array}
} \\
(A, X) = \left( \frac{\bar{X}}{2}, \frac{\bar{X}}{2} \right) \\
U_{EQ1} = \frac{(\bar{X})^2}{4}
\end{array}
} \qquad (13)
$$

| high wage competitive equilibrium,it exists iff $\boxed{\bar{X} > 0}$ |
| --- |
| $w = 2$ |
| $(A_2, X_2, \Pi_2) = (\bar{X}, \frac{\bar{X}}{2}, 0)$ |
| $(A_1, X_1, \Pi_1) = (0,0,0)$ |
| $(A, X) = \left(\bar{X}, \frac{\bar{X}}{2}\right)$ |
| $U_{EQ2} = \frac{(\bar{X})^2}{2}$ |

(14)

Comparing (14) to (13) we observe that there exist two equilibria iff $\bar{X} \geq 2$ ,and a unique equilibrium iff $0 < \bar{X} < 2$

To understand the multiplicity of equilibria, we compute and draw the aggregate input demand correspondence $X_D = X_1 + X_2$ and net input supply correspondence $\bar{X} - X$ .

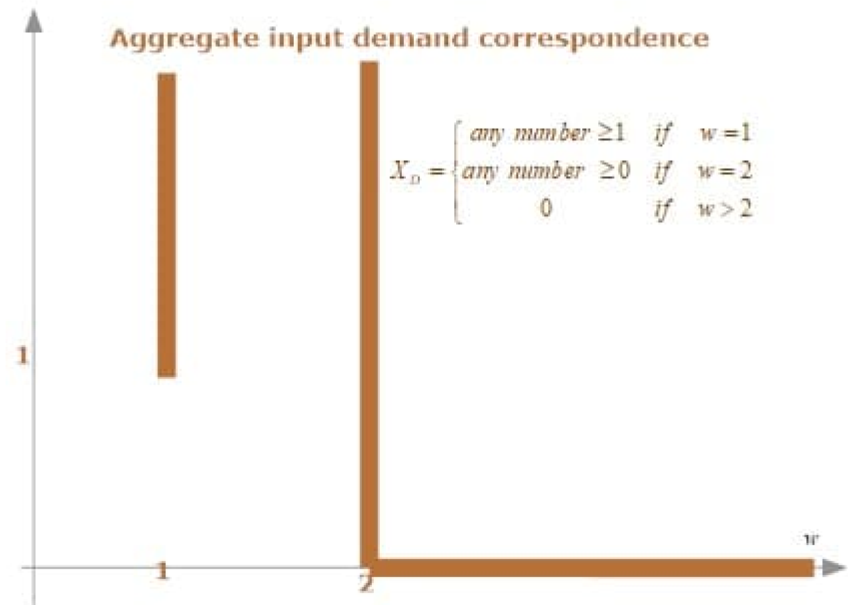
Computation of the aggregate input demand correspondence $X_D = X_1 + X_2$ as a function of w alone

By (5),(6) we obtain

- If w<1 then $X_1$ ,and therefore $X_D$, are not defined.
- If $1 < w < 2$ then $X_1 = 0, \frac{2}{1+X_1} = 2 > w,$ and therefore $X_2$ and $X_D$ are not defined.
- If $w > 2$ then $w > \frac{2}{1+X_1}$ and therefore $X_D = 0$
- If $w = 1$ then $w \geq \frac{2}{1+X_1}$ iff $X_1 \geq 1$ and therefore

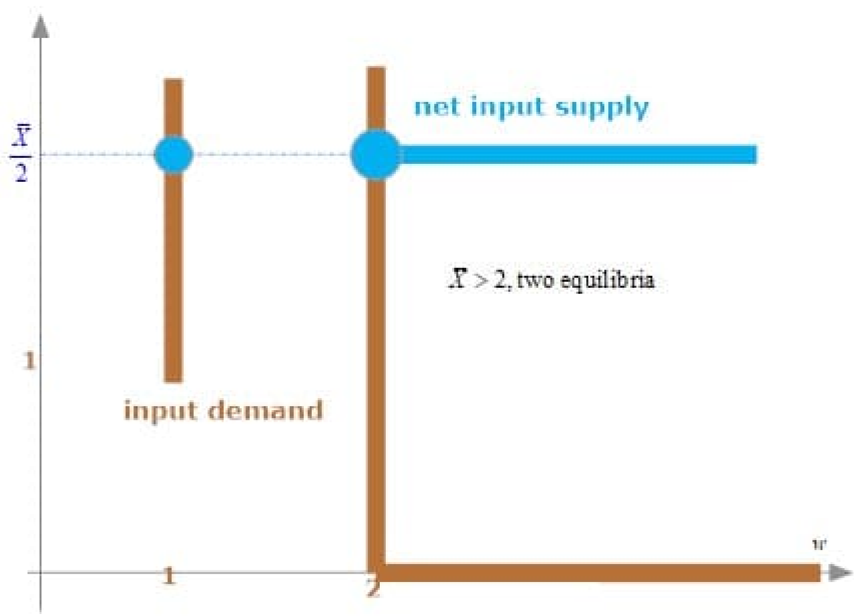
$$X_D = \begin{cases} 1 + X_2, \text{any } X_2 \geq 0 & \text{if} \quad X_1 = 1 \\ (\text{any})\, X_1 & \text{if} \quad X_1 > 1 \end{cases} = \text{any number } \geq 1$$

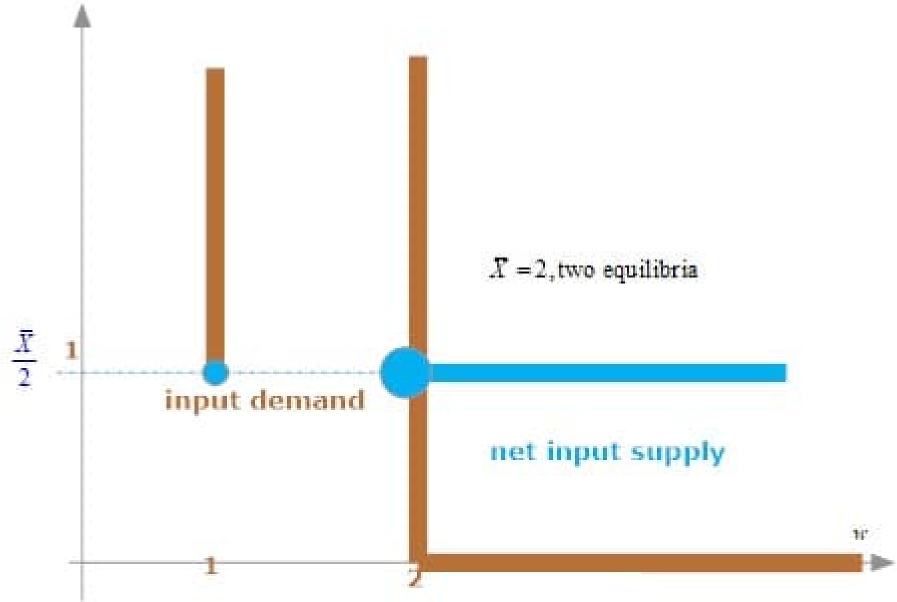
- If $w = 2$ then $X_1 = 0, X_D = X_2 = \text{any number } \geq 0$ .

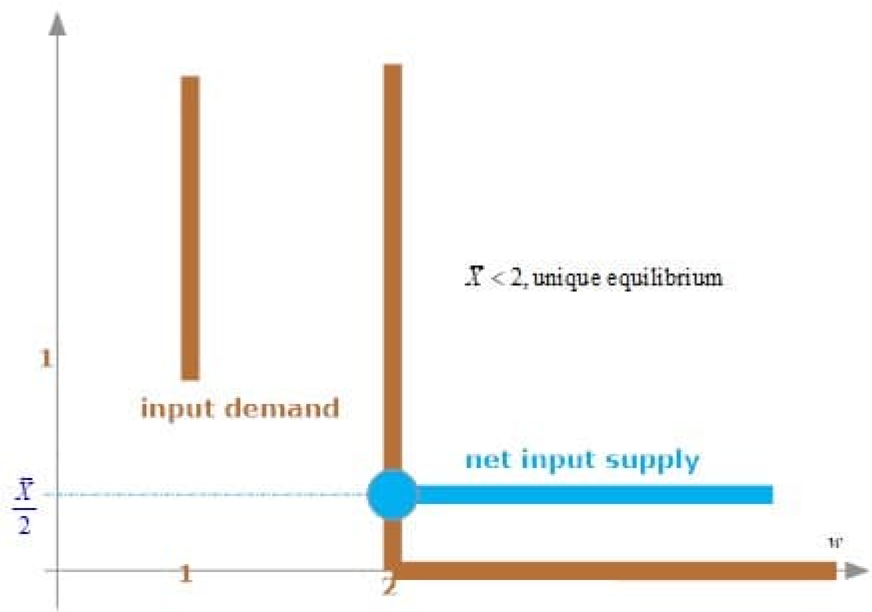



Computation of the aggregate input net supply $X_S = \overline{X} - X$ at values of $w$ consistent with (9)

By (3),(9),(4) $X = \frac{\overline{X}}{2}, X_S = \frac{\overline{X}}{2}$ at any such w; when w=1 then $X, X_S$ are defined iff $X_1 \geq 1$






## WELFARE COMPARISONS

We compute the best allocations for the consumer, and then compare it to the equilibrium allocations.

pareto efficiency problem with one consumer

max $U = AX$

subject to

resource constraints $A \le A_1 + A_2, X + X_1 + X_2 \le \bar{X}$

technology constraints $A_1 = X_1, A_2 = \dfrac{2X_2}{1+X_1}, A_1 \ge 0, A_2 \ge 0, X_1 \ge 0, X_2 \ge 0$ (15)

variables $A_1, A_2, X_1, X_2, A, X$ (the allocation of resources)

parameters $\bar{X}$

conditions on parameters $\bar{X} > 0$

Pareto efficient allocation

$$\left(A_1, X_1\right) = (0,0), \left(A_2, X_2\right) = (\bar{X}, \frac{\bar{X}}{2}) = \left(A, X\right) \qquad (16)$$

$$U^{MAX} = \frac{\left(\bar{X}\right)^2}{2}$$

Comparing the efficient allocation (16) to the equilibrium allocations (13),(14) we observe that the low wage equilibrium is inefficient, while the high wage equilibrium is efficient. For $\bar{X} \ge 2$ ,this economy has two equilibria that are pareto ranked, in the sense that $U^{MAX} = U_{EQ2} > U_{EQ1}$ .