0. THE ECONOMY

1. Consumers: 1, 2

2. Goods: A, L

3. Preferences/endowments

$$U_1 = A_1 - \frac{1}{2}L^2{}_1, e_1 = (0,1)$$
$$U_2 = A_2, e_2 = (0,0)$$

Player 2 receives all profit income.

4. Firms: 1, 2... n

5. Production function (the same for all firms): $\hat{A} = F(\hat{L})$

6. Equilibrium conditions

demand	=	supply
$A_1 + A_2$	=	$\sum_{i=1}^{n} \hat{A}_i$
$\sum_{i=1}^{n} \hat{L}_i$	=	L_1

We will examine three types of production function: decreasing, constant, increasing turns to scale; and three kinds of equilibria: competitive, monopolistic, Cournot.

1. CONSTANT RETURNS TO SCALE $\hat{A} = \hat{L}$

PARETO POINTS are the solutions of

$$\max U_1 = A_1 - \frac{1}{2}L^2{}_1, \text{subject to } U_2 = A_2 \geq \bar{U}_2, A_1 + A_2 \leq \sum_{j=1}^{n} \hat{A}_j, \hat{A}_j = \hat{L}_j, \sum_{j=1}^{n} \hat{L}_j \leq L_1, 0 \leq L_1 \leq 1$$

$$\boxed{L_1 = 1, A_1 + A_2 = 1, \sum_{j=1}^{n} \hat{A}_j = 1 = \sum_{j=1}^{n} \hat{L}_j, U_1 + U_2 = \frac{1}{2}, 0 \leq U_2 \leq 1} \qquad (1.1)$$

A.COMPETITIVE EQUILIBRIUM

1. Prices: p for A, w for L.

2. Normalization p=1

3. Incomes

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$$M_1 = wL_1, M_2 = \sum_{j=1}^{n} \Pi_j \tag{1.2}$$

4. Consumer optimization problems

$$\text{max } U_1 = A_1 - \frac{1}{2}L^2{}_1$$
$$\text{subject to } 0 \le A_1 \le M_1, 0 \le L_1 \le 1$$

$$\text{max } U_2 = A_2$$
$$\text{subject to } 0 \le A_2 \le M_2$$

$$(A_1, L_1) = \begin{cases} (w^2, w) & \text{if } w \le 1 \\ (w, 1) & \text{if } w \ge 1 \end{cases} \tag{1.3}$$

$$A_2 = M_2 \tag{1.4}$$

5. Equilibrium conditions

$$\begin{array}{ccc} \text{demand} & = & \text{supply} \\ A_1 + A_2 & = & \sum_{i=1}^{n} \hat{A}_i \\ \sum_{i=1}^{n} \hat{L}_i & = & L_1 \end{array} \tag{1.5}$$

6. Firm optimization problems

Each firm j takes prices and everything else except its own inputs/outputs as given, and chooses input demand $\hat{L}_j$ so as to

$$\text{max } \Pi_j = p\hat{A}_j - w\hat{L}_j = (1-w)\hat{L}_j$$

For j=1,…,n

$$\hat{A}_j = \hat{L}_j = \begin{cases} \infty & \text{if } w<1 \\ \ge 0 & \text{if } w=1 \\ 0 & \text{if } w>1 \end{cases} \tag{1.6}$$

7. Competitive equilibrium prices, quantities, utilities

$$w = 1, \Pi_j = 0 \tag{1.7}$$

$$\sum_{j=1}^{n} \hat{L}_j = 1 = \sum_{j=1}^{n} \hat{A}_j$$
$$A_1 = L_1 = 1, U_2^E = \frac{1}{2} \quad (1.8)$$
$$A_2 = 0 = U_2^E$$

competitive equilibria are pareto efficient

B.COURNOT EQUILIBRIUM

Steps 1,..,5 are the same

6. Firm optimization problems. Each firm j chooses its own inputs/outputs, i.e $\hat{L}_j$,to maximize profits, taking as given the inputs/outputs of other firms, but not prices.

$$\max \ \Pi_j = \hat{A}_j - w\hat{L}_j = (1-w)\hat{L}_j$$

subject to

$$\sum_{i=1}^{n} \hat{L}_i = L_1$$

$$L_1 = \begin{cases} w & \text{if } w \le 1 \\ 1 & \text{if } w \ge 1 \end{cases}$$

This problem becomes

$$\max \ \Pi_j = (1 - \sum_{i=1}^{n} \hat{L}_i)\hat{L}_j, \text{subject to } \sum_{i=1}^{n} \hat{L}_i \le 1$$

$$\hat{A}_j = \hat{L}_j = \frac{1}{1+n} \quad (1.9)$$

7. Cournot equilibrium prices, quantities, utilities

$$w = \frac{n}{n+1}, \Pi_j = (\frac{1}{n+1})^2 \quad (1.10)$$

$$L_1 = \frac{n}{n+1}, A_1 = (\frac{n}{n+1})^2, U^C{}_1 = \frac{1}{2}(\frac{n}{n+1})^2$$
$$A_2 = (\frac{1}{n+1})^2 = U^C{}_2 \quad (1.11)$$

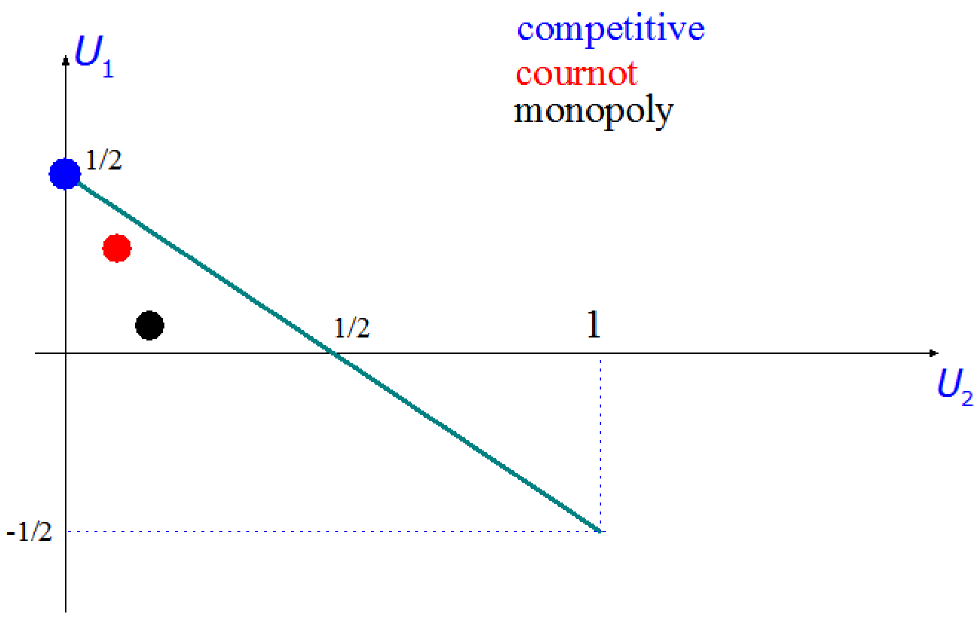
Cournot equilibria are inefficient. They tend to competitive equilibria, hence to efficiency, as $n \to \infty$

C.MONOPOLISTIC EQUILIBRIUM

Cournot with n=1.

$$w = \frac{1}{2}, \Pi_j = \frac{1}{4} \tag{1.12}$$

$$L_1 = \frac{1}{2}, A_1 = \frac{1}{4}, U^M{}_1 = \frac{1}{8}$$
$$A_2 = \frac{1}{4} = U^M{}_2 \tag{1.13}$$



2. DECREASING RETURNS TO SCALE, $\hat{A} = 2\sqrt{\hat{L}}$

PARETO POINTS

are the solutions of

$$\max U_1 = A_1 - \frac{1}{2}L^2{}_1, \text{subject to } U_2 = A_2 \geq \bar{U}_2, A_1 + A_2 \leq \sum_{j=1}^{n} \hat{A}_j, \hat{A}_j = 2\sqrt{\hat{L}_j}, , \sum_{j=1}^{n} \hat{L}_j \leq L_1, 0 \leq L_1 \leq 1$$

The solutions are

$$L_1 = 1, \hat{L}_j = \frac{1}{n}, \hat{A}_j = \frac{2}{\sqrt{n}}, A_1 + A_2 = 2\sqrt{n}, U_1 + U_2 = 2\sqrt{n} - \frac{1}{2}, 0 \leq U_2 \leq 2\sqrt{n} \tag{2.1}$$

A.COMPETITIVE EQUILIBRIUM

Steps 1,..,5 are the same as before

6.firm optimization problems

Each firm j takes prices and everything else except its own inputs/outputs as given, and chooses input demand $\hat{L}_j$ so as to $\max\ \Pi_j = p\hat{A}_j - w\hat{L}_j = 2\sqrt{\hat{L}_j} - w\hat{L}_j$.For j=1,...,n

$$\hat{L}_j = \frac{1}{w^2}, \hat{A}_j = \frac{2}{w}, \Pi_j = \frac{1}{w} \tag{2.2}$$

7.competitive equilibrium prices, quantities,utilities

$$w = \sqrt{n}, \Pi_j = \frac{1}{\sqrt{n}} \tag{2.3}$$

$$\begin{aligned} &\hat{L}_j = \frac{1}{n}, \hat{A}_j = \frac{2}{\sqrt{n}} \\ &A_1 = \sqrt{n}, L_1 = 1, U^E{}_1 = \sqrt{n} - \frac{1}{2} \\ &A_2 = \sqrt{n} = U^E{}_2 \end{aligned} \tag{2.4}$$

competitive equilibria are pareto efficient

B.COURNOT EQUILIBRIUM

Steps 1,..,5 are the same

6.firm optimization problem.Each firm j chooses its own inputs/outputs,i.e $\hat{L}_j$, to maximize profits,taking as given the inputs/outputs of other firms,but not prices.

$$\max\ \Pi_j = \hat{A}_j - w\hat{L}_j = 2\sqrt{\hat{L}_j} - w\hat{L}_j$$

$$\text{subject to } \sum_{i=1}^{n} \hat{L}_i = L_1$$

$$L_1 = \begin{cases} w & \text{if } w \le 1 \\ 1 & \text{if } w \ge 1 \end{cases}$$

This problem becomes

$$\max\ \Pi_j = 2\sqrt{\hat{L}_j} - \Big(\sum_{i=1, i\neq j}^{n} \hat{L}_i + \hat{L}_j\Big)\hat{L}_j$$

$$\text{subject to } \sum_{i=1, i\neq j}^{n} \hat{L}_i + \hat{L}_j \le 1$$

$$\text{where } \sum_{i=1}^{n} \hat{L}_i = \sum_{i=1, i\neq j}^{n} \hat{L}_i + \hat{L}_j$$

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Note that all firms face the same constraint.For j=1,...,n

$$\hat{L}_j = \begin{cases} \dfrac{1}{(n+1)^{\frac{2}{3}}} & \text{if} \quad n = 1,2 \\ \dfrac{1}{n} & \text{if} \quad n \geq 3 \end{cases}$$

$$\hat{A}_j = \begin{cases} \dfrac{2}{(n+1)^{\frac{1}{3}}} & \text{if} \quad n = 1,2 \\ \dfrac{2}{\sqrt{n}} & \text{if} \quad n \geq 3 \end{cases} \tag{2.5}$$

$$w = \begin{cases} \dfrac{n}{(n+1)^{\frac{2}{3}}} & \text{if} \quad n = 1,2 \\ 1 & \text{if} \quad n \geq 3 \end{cases}$$

$$\Pi_j = \begin{cases} \dfrac{2}{(n+1)^{\frac{1}{3}}} - \dfrac{n}{(n+1)^{\frac{4}{3}}} & \text{if} \quad n = 1,2 \\ \dfrac{2}{\sqrt{n}} - \dfrac{1}{n} & \text{if} \quad n \geq 3 \end{cases} \tag{2.6}$$

7.equilibrium prices, quantities,utilities

$$L_1 = \begin{cases} \dfrac{n}{(n+1)^{\frac{2}{3}}} & \text{if} \quad n = 1,2 \\ 1 & \text{if} \quad n \geq 3 \end{cases}$$

$$A_1 = \begin{cases} \dfrac{n^2}{(n+1)^{\frac{4}{3}}} & \text{if} \quad n = 1,2 \\ 1 & \text{if} \quad n \geq 3 \end{cases} \tag{2.7}$$

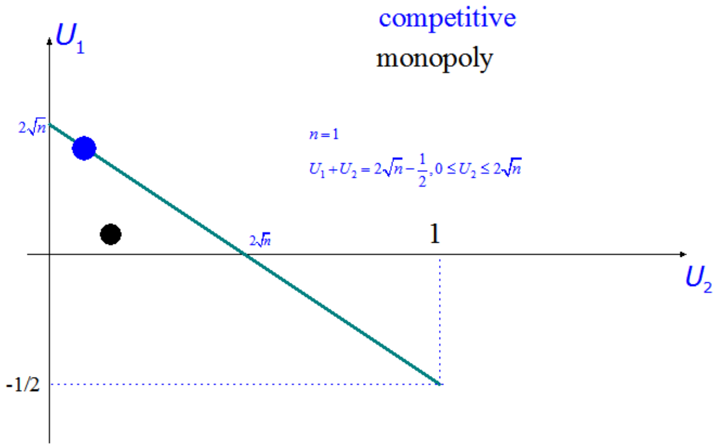
$$A_2 = U_2 = \begin{cases} \dfrac{2n}{(n+1)^{\frac{1}{3}}} - \dfrac{n^2}{(n+1)^{\frac{4}{3}}} & \text{if} \quad n = 1,2 \\ 2\sqrt{n} - 1 & \text{if} \quad n \geq 3 \end{cases}$$

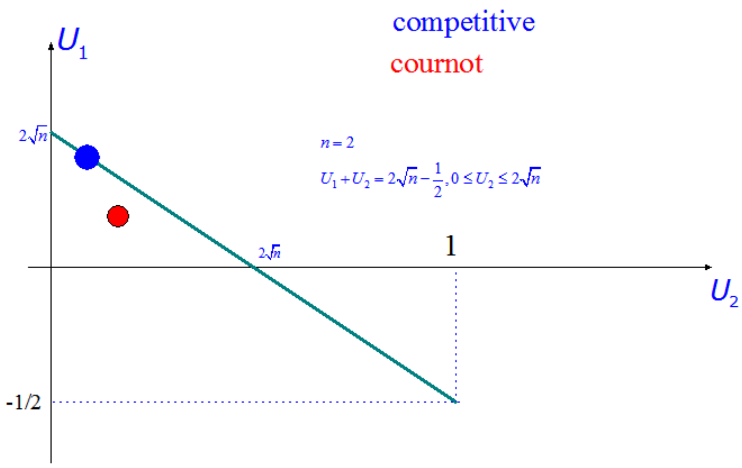
$$U_1 = \begin{cases} \dfrac{n^2}{2(n+1)^{\frac{4}{3}}} & \text{if} \quad n = 1,2 \\ \dfrac{1}{2} & \text{if} \quad n \geq 3 \end{cases} \tag{2.8}$$

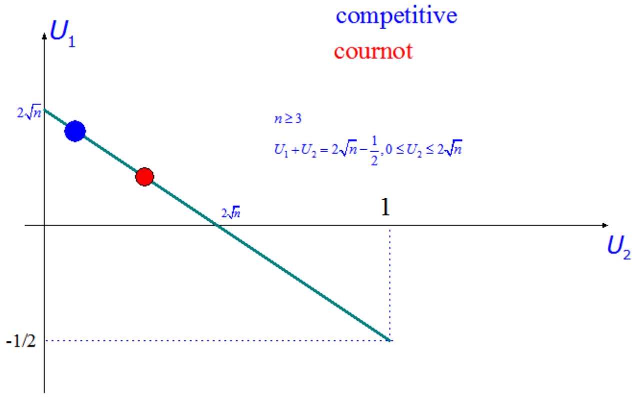
Cournot equilibria are inefficient when n<3, efficient otherwise

C.MONOPOLISTIC EQUILIBRIUM

Cournot with n=1







3. INCREASING RETURNS TO SCALE, $\hat{A}=\max(\hat{L}-f,0)$, $0<f<\frac{1}{2}$

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PARETO POINTS are the solutions of

$$\max U_1 = A_1 - \frac{1}{2}L^2_1, \text{subject to } U_2 = A_2 \geq \bar{U}_2, A_1 + A_2 \leq \sum_{j=1}^{n} \hat{A}_j, \hat{A}_j = \max(\hat{L}_j - f, 0),, \sum_{j=1}^{n} \hat{L}_j \leq L_1, 0 \leq L_1 \leq 1$$

Only one firm will be active,say firm 1.

$$\hat{L}_1 = 1, \hat{A}_1 = 1 - f, \hat{L}_j = 0, \hat{A}_j = 0, j = 2..n, L_1 = 1, A_1 + A_2 = 1 - f, U_1 + U_2 = \frac{1}{2} - f, 0 \leq U_2 \leq 1 - f$$

(2.9)

A.COMPETITIVE EQUILIBRIUM

Steps 1,..,5 are the same as before

6. Firm optimization problems. Each firm j takes prices and everything else except its own inputs/outputs as given, and chooses input demand $\hat{L}_j$ so as to

$$\max \ \Pi_j = p\hat{A}_j - w\hat{L}_j = \begin{cases} -w\hat{L}_j & if \quad \hat{L}_j < f \\ (1-w)\hat{L}_j - f & if \quad \hat{L}_j \geq f \end{cases}$$

For j=1,…,n

$$\hat{L}_j = \begin{cases} 0 & if \quad w \leq 1 \\ \infty & if \quad w > 1 \end{cases} \qquad (2.10)$$

3.3 is incompatible with 1.4,1.6 hence

There is no competitive equilibrium under increasing returns to scale

B.COURNOT EQUILIBRIUM

Steps 1... 5 are the same

6. Firm optimization problems. Each firm j chooses its own inputs/outputs,i.e $\hat{L}_j$, to maximize profits, taking as given the inputs/outputs of other firms, but not prices.

$$\max \ \Pi_j = \hat{A}_j - w\hat{L}_j = \begin{cases} -w\hat{L}_j & if \quad \hat{L}_j < f \\ (1-w)\hat{L}_j - f & if \quad \hat{L}_j \geq f \end{cases}$$

$$\text{subject to } \sum_{i=1}^{n} \hat{L}_i = L_1$$

$$L_1 = \begin{cases} w & \text{if} & w \leq 1 \\ 1 & \text{if} & w \geq 1 \end{cases}$$

This problem becomes

$$\max \ \Pi_j = \begin{cases} -(\sum_{i=1,i\neq j}^{n} \hat{L}_i + \hat{L}_j)\hat{L}_j & if \quad \hat{L}_j < f \\ (1-(\sum_{i=1,i\neq j}^{n} \hat{L}_i + \hat{L}_j))\hat{L}_j - f & if \quad \hat{L}_j \geq f \end{cases}$$

$$\text{subject to} \ \sum_{i=1,i\neq j}^{n} \hat{L}_i + \hat{L}_j \leq 1$$

$$\text{where} \ \sum_{i=1}^{n} \hat{L}_i = \sum_{i=1,i\neq j}^{n} \hat{L}_i + \hat{L}_j$$

Note that all firms face the same constraint.Let 1,2,..,m be the active firms.If $m \geq 1$ then for j=1,...,m

$$\hat{L}_j = \frac{1}{m+1}, \hat{A}_j = \frac{1}{m+1} - f, w = \frac{m}{m+1} \qquad (2.11)$$
$$\Pi_j = \frac{1}{(m+1)^2} - f$$

The condition $\Pi_j \geq 0$ yields

$$m \leq \frac{1}{\sqrt{f}} - 1 \qquad (2.12)$$

By 3.5,

$$m \geq 1 \Leftrightarrow f \leq \frac{1}{4} \qquad (2.13)$$

7. Equilibrium prices, quantities, utilities

$$case1: \frac{1}{4} < f \leq \frac{1}{2} \Rightarrow \text{all quantities are zero} \qquad (2.14)$$

Cournot equilibria are inefficient

$$case2: f \leq \frac{1}{4} \Rightarrow \qquad (2.15)$$
$$m \leq \frac{1}{\sqrt{f}} - 1, A_1 = (\frac{m}{m+1})^2, L_1 = \frac{m}{m+1}, U_1 = \frac{1}{2}(\frac{m}{m+1})^2, U_2 = A_2 = m((\frac{1}{m+1})^2 - f)$$

Cournot equilibria are inefficient.

The case $\frac{1}{9} < f \leq \frac{1}{4}$ *implies m=1 and is called a natural monopoly*

In general, if we have free entry and exit of firms, the number of firms is a variable determined by the conditions $\Pi_j(m) \geq 0, \Pi_j(m+1) \leq 0$ .By (2.11),(2.12) this implies $m = \left\lfloor \frac{1}{\sqrt{f}} - 1 \right\rfloor$ .The value of m can be any integer between 0 and infinity. We speak of a natural monopoly, duopoly,...as m=1,m=2,..

Cournot equilibria are inefficient for two reasons

1. Underproduction and underemployment, because firms raise prices by limiting output.

2. Excessive competition and hence duplication of fixed costs, because firms enter until profits are competed away.