$X =$ choice set, $X = R_+^L$ in consumer theory

F = some family of subsets of X.

F = all sets $B = \{x \in X : px \leq pe\}$ for some $p \gg 0$, $e \geq 0$ in consumer theory

A choice function is a function $J : F \to X$ with $J(B) \in B \quad \forall B \in F$

$J(p)$ = demand function, or excess demand function, in consumer theory

$\boxed{\text{$J$ is RATIONAL if there exists a relation $G \subset X \times X$, complete, transitive, and acyclic, such that $\forall B \in F$, $\forall x \in B$: $J(B)\, G\, x$.}}$

G is a preference relation in consumer theory. We say that such a G rationalizes J.

Main question: characterize all rational choice functions intrinsically, ie without reference to G.

Example 1 $\quad X = \{\alpha, \beta, \gamma, \delta, \epsilon\}$

$F = \{\alpha\beta\gamma\delta, \alpha\gamma\delta\epsilon, \beta\gamma\delta, \alpha\epsilon\}$

$G = \delta\gamma\alpha\beta\epsilon$ (δ is the top choice, γ the second best choice, etc...)

G as a relation is then written as

$$G = \{\delta\alpha, \delta\beta, \delta\gamma, \delta\epsilon, \gamma\alpha, \gamma\beta, \gamma\epsilon, \alpha\beta, \alpha\epsilon\}$$

Each choice function J defines a relation on F

REVEALED PREFERENCE ON FEASIBLE SETS

$$\boxed{B \longrightarrow C \quad \text{iff} \quad J(B) \neq J(C) \quad \text{and} \quad J(C) \in B}$$

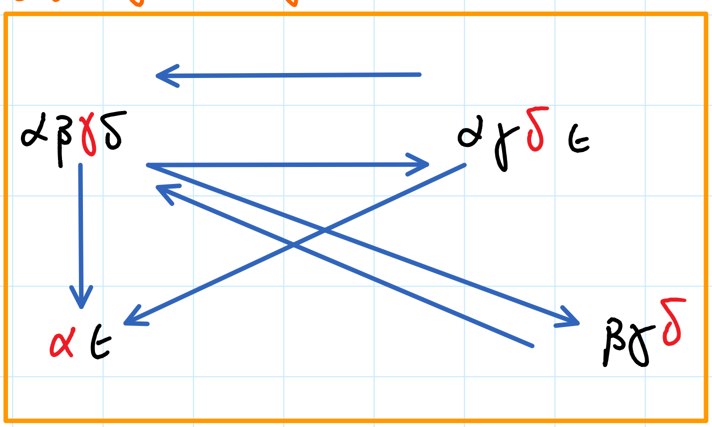
$J(C)$ is available at B, but J chooses $J(B) \neq J(C)$.

Example 1 Consider the choice function $J : F \to X$, given by

$J(\alpha\beta\gamma\delta) = \gamma$; $J(\alpha\gamma\delta\epsilon) = \delta$, $J(\beta\gamma\delta) = \delta$, $J(\alpha\epsilon) = \alpha$

Then J induces the following partial order on F

Choice function J



and the induced relation on F

Red letters are targets of arrows from sets that include the same letter in black. No other arrows exist.

Lemma 1: If J is rational, then the induced relation on F has no cycles of length 0 or 1.

Proof: $B \to B$ is impossible, hence $\to$ has no cycles of length 0.

If $\to$ has cycles of length 1, then there exist B, C in $\mathcal{F}$ such that

$$B \rightleftarrows C$$

ie $\mathcal{J}(B) \neq \mathcal{J}(C)$, $\mathcal{J}(C) \in B$, $\mathcal{J}(B) \in C$. Let G be the relation that rationalizes $\mathcal{J}$. Then

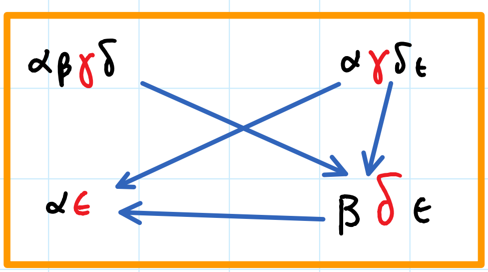
$\mathcal{J}(C) \in B \Rightarrow \mathcal{J}(B)\ G\ \mathcal{J}(C)$ | This contradicts the asymmetry
$\mathcal{J}(B) \in C \Rightarrow \mathcal{J}(C)\ G\ \mathcal{J}(B)$ | of G

Definition: $\mathcal{J}$ satisfies the $\overbrace{\text{weak axiom of revealed preference}}^{\text{WARP}}$ if its induced relation on feasible sets has no cycles of length zero or one.

Lemma 1 $\boxed{\text{If } \mathcal{J} \text{ is rational, then } \mathcal{J} \text{ satisfies WARP}}$

Example 1: $\mathcal{J}$ is not rational, because its induced relation has two cycles of length 1. The choice function h, given by the diagram below, is rational

Rational choice function h



and induced relation on F

Lemma 2: If $\mathcal{J}$ is rational then the induced relation on F has no cycles of any length.

Proof: Suppose, for contradiction, that $\mathcal{J}$ is rational and that its induced relation on F has a cycle of length $N \geq 2$,

$B = B^1 \to B^2 \to \dots \to B^N = B$. Then

$\mathcal{J}(B^i) \neq \mathcal{J}(B^{i+1})$, $\mathcal{J}(B^{i+1}) \in B^i$ $\forall i = 1 \dots N-1$, ie

$\mathcal{J}(B) \neq \mathcal{J}(B^2) \neq \dots \neq \mathcal{J}(B^{N-1}) \neq \mathcal{J}(B)$, and

$\mathcal{J}(B^2) \in B$, $\mathcal{J}(B^3) \in B^2, \dots, \mathcal{J}(B^{N-1}) \in B^{N-2}$, $\mathcal{J}(B) \in B^{N-1}$

Since $\mathcal{J}$ is rationalized by G, we obtain

$$\begin{aligned}
&\mathcal{J}(B)\ G\ \mathcal{J}(B^2) \\
&\mathcal{J}(B^2)\ G\ \mathcal{J}(B^3) \\
&\quad\vdots \\
&\mathcal{J}(B^{N-2})\ G\ \mathcal{J}(B^{N-1}) \\
&\mathcal{J}(B^{N-1})\ G\ \mathcal{J}(B)
\end{aligned}
\quad \overset{\text{by}}{\Longrightarrow} \quad \mathcal{J}(B)\ G\ \mathcal{J}(B),$$

transitivity of G

contradiction because G is irreflexive

Definition: A choice function $\mathcal{J}$ satisfies the strong axiom of revealed preference (SARP) if the induced relation on F has no cycles of any length.

Lemma 2: $\boxed{\text{If } \mathcal{J} \text{ is rational, then } \mathcal{J} \text{ satisfies SARP}}$

Theorem: $\boxed{\mathcal{J} \text{ is rational iff it satisfies SARP}}$

Proof: By lemma 2, if $\mathcal{J}$ is rational then it satisfies SARP.

Suppose that $\mathcal{J}$ satisfies SARP. Show that $\mathcal{J}$ is rational

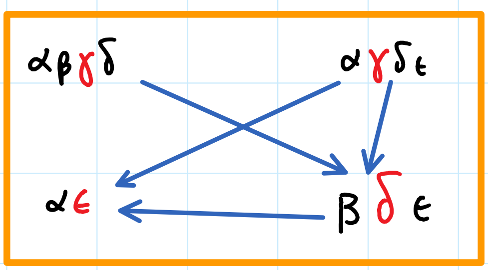
Define a new relation $\Rightarrow$ on X by

$x \Rightarrow y$ iff $x \neq y$ and there is $B \in \mathcal{F}$, $x = \mathcal{J}(B)$, $y \in B$.

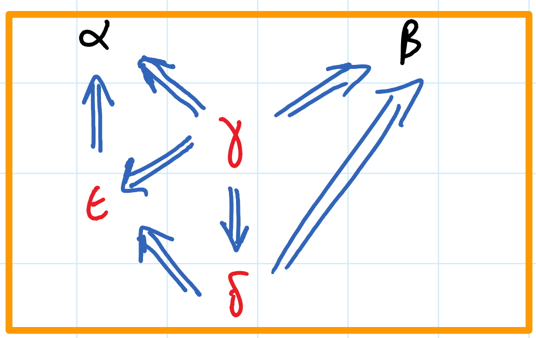
(At B, y was available but x was chosen by $\mathcal{J}$)

Note that the choice function h of example 1 induces the following relation $\Rightarrow$ on X

Choice function h and induced relation on F



Induced relation $\Rightarrow$ on X



Red letters point to any other letter that belongs together with red in some $B \in \mathcal{F}$

$\Rightarrow$ is acyclic, for if not, there exists a path in X

(hence irreflexive, asymmetric)

$$x = x^1 \Rightarrow x^2 \Rightarrow \dots \Rightarrow x^{N-1} \Rightarrow x^N = x$$

Hence $x = x^1 \neq x^2 \neq \dots \neq x^{N-1} \neq x^N = x$, and there exist $B^i \in F$ such that $x^i = y(B^i)$, $x^{i+1} \in B^i$, $\forall i = 1, \dots, N-1$. Hence $y(B^i) \neq y(B^{i+1})$ $\forall i = 1 \dots N-1$, $y(B^{i+1}) \in B^i$, ie

$$B^1 \to B^2 \to \dots \to B^{N-1}$$

Finally $x = x^1 = y(B^1)$ and $x = x^N \in B^{N-1}$ ie

$y(B^1) \in B^{N-1}$. Then if $y(B^1) \neq y(B^{N-1})$ we must conclude that $B^{N-1} \to B^1$, contradicting SARP.

Hence $y(B^1) = y(B^{N-1})$, ie $x = x^1 = x^{N-1}$. But $x^N = x$ implies $x^{N-1} = x^N$, a contradiction.

$\Rightarrow$ may not be complete or transitive

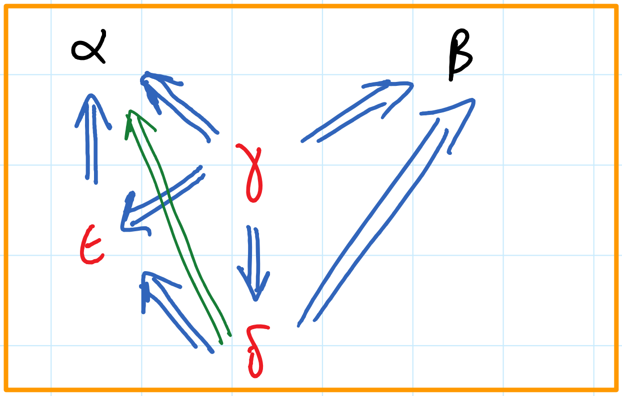
In the example, $\Rightarrow$ is not transitive because it misses the arrow $\delta \Rightarrow \alpha$, and it is not complete because there is no arrow between α and β.

Extend $\Rightarrow$ to its transitive closure $\overset{*}{\Rightarrow}$, by filling in triangles. Hence $\overset{*}{\Rightarrow}$ satisfies

$x \Rightarrow y$ implies $x \overset{*}{\Rightarrow} y$, and

$\overset{*}{\Rightarrow}$ is acyclic and transitive

$\Rightarrow$ is acyclic and transitive



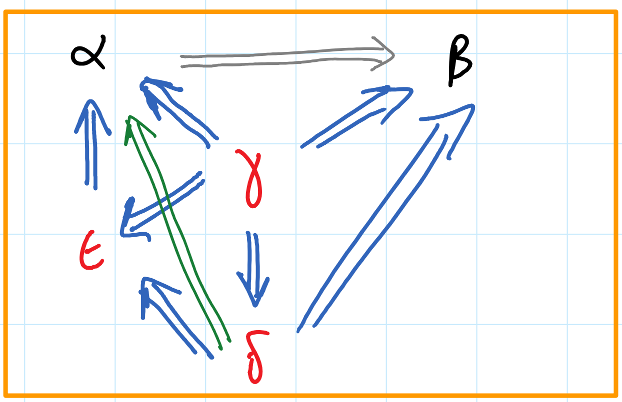
The transitive closure $\overset{*}{\Rightarrow}$ of $\Rightarrow$

Extend $\overset{*}{\Rightarrow}$ to its complete closure G

(Topological sorting if X is finite, Zorn otherwise). Hence G satisfies

$x \Rightarrow y$ implies $x \, G \, y$

G is acyclic, complete and transitive



The complete transitive closure G of $\Rightarrow$

Show that G rationalizes $\mathcal{J}$

Let $y \in B$, $y \neq \mathcal{J}(B)$. Then $\mathcal{J}(B) \Rightarrow y$, hence

$\mathcal{J}(B) \, G \, y$ •–

Proposition: If F contains all 3-element subsets of X, then $\mathcal{J}$ satisfies WARP iff $\mathcal{J}$ satisfies SARP

Proof Suppose $\mathcal{J}$ satisfies WARP. We show that the induced relation $\to$ on F is transitive. Let $B \to C \to D$. Then $\mathcal{J}(B) \neq \mathcal{J}(C) \neq \mathcal{J}(D)$, $\mathcal{J}(C) \in B$, $\mathcal{J}(D) \in C$. We need to show that $B \to D$ ie $\mathcal{J}(B) \neq \mathcal{J}(D)$ and $\mathcal{J}(D) \in B$.

OR

$\mathcal{J}(B) = \mathcal{J}(D)$

Then $\mathcal{J}(C) \in B$

$\mathcal{J}(B) \in C$

$$\frac{\mathcal{J}(B) \neq \mathcal{J}(C)}{B \to C \to B}$$

contradicting WARP

$\mathcal{J}(B) \neq \mathcal{J}(D)$

Let $E = \{\mathcal{J}(B), \mathcal{J}(C), \mathcal{J}(D)\}$.

If $\mathcal{J}_E = \mathcal{J}_C$ then

$$B \rightleftarrows E$$, contradicting WARP

If $\mathcal{J}_E = \mathcal{J}_D$ then

$$C \rightleftarrows E$$, contradicting WARP

Hence $\mathcal{J}_E = \mathcal{J}_B$, and

$$\begin{tikzcd} B \arrow[r] & C \arrow[r] & D \\ & E \arrow[u] \arrow[ur] & \end{tikzcd}$$

If $\mathcal{J}_D \notin B$ then $\mathcal{J}_D \neq \mathcal{J}_E$, hence $D \to E$

Contradicting WARP. Hence $y_D \in B$.

But then $B \to D$, QED

Consumption set R^L_+.

A consumer is a pair (C, e), where

$e \in R^L_+$

C is a complete and transitive relation on R^L_+.

The consumer problem at prices $p \gg 0$ is

$$\boxed{\begin{array}{ll} \max C & \text{Objective relation} \\ \text{subject to } py \leq pe,\ y \geq 0 & \text{Feasible set} \\ \text{Variables: } y\ ;\ \text{Parameters: } p, e. & \end{array}} \qquad (M_p)$$

A solution to the consumer's problem at p is a point $\hat{y}$ that is feasible, and that satisfies $\hat{y} C y$ for any other feasible point y

We will only consider preferences C such that $\forall p \gg 0$ the solution $\hat{y}$ is unique and exhausts income $(p\hat{y} = pe)$

The solution $\hat{y}$ at p is denoted by $y(p)$

The excess demand of consumer (C, e) at p is the vector $\mathcal{J}(p) = y(p) - e$. The function $\mathcal{J}$ is called an individual excess demand function. Every such function satisfies, for all $p \gg 0$

$$\mathcal{J}(\lambda p) = \lambda \mathcal{J}(p), \ \forall \lambda > 0 \qquad \text{Zero-homogeneity} \qquad H$$

$$p\mathcal{J}(p) = 0 \qquad \text{WALRAS LAW} \qquad W$$

$$\mathcal{J}(p) + b \gg 0, \text{ for some } b \gg 0 \qquad \text{Bounded from below} \qquad B$$

Example 1 For the consumer with $U(y_1, y_2) = y_1 y_2$ and $e = (1, 0)$

$$y(p) = \begin{pmatrix} 1/2 \\ \dfrac{p_1}{2p_2} \end{pmatrix}, \quad \mathcal{J}(p) = \begin{pmatrix} -1/2 \\ \dfrac{p_1}{2p_2} \end{pmatrix}$$

H, W are obvious by calculation. Take $b = \binom{1}{1}$

A function $\mathcal{J}: R^L_{++} \to R^L$ is RATIONAL if there exists a consumer (C, e) whose excess demand function is $\mathcal{J}$

Lemma 1: If $\mathcal{J}$ is rational then it satisfies H-W-B

Is this all? To answer this, introduce a relation on prices

Each function $\mathcal{J} : R^L_{++} \to R^L$ induces a relation on prices

REVEALED PREFERENCE

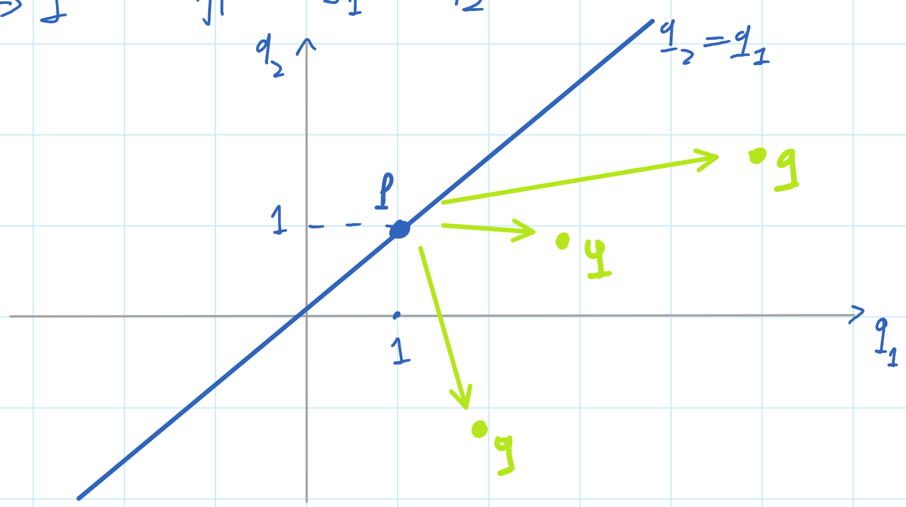
$$\boxed{p \longrightarrow q \quad \text{iff} \quad \mathcal{J}(p) \neq \mathcal{J}(q) \ \text{and} \ p\mathcal{J}(q) \leq 0}$$

Reasoning: At p I could buy $\mathcal{J}(q)$ but I chose $\mathcal{J}(p)$

Example 1: Let $p = (1,1)$. Then

$p\mathcal{J}(q) = -\frac{1}{2} + \frac{q_1}{2q_2} \leq 0$ iff $q_1 \leq q_2$. Hence

$p \longrightarrow q$ iff $q_1 < q_2$



$p = (1,1)$ is revealed preferred to every q below the line $q_2 = q_1$.

WEAK AXIOM OF REVEALED PREFERENCE WARP

$\mathcal{J}$ satisfies WARP if the induced relation $\longrightarrow$ on prices has no cycles of length zero or one, i.e. if $p \to p$ is impossible (it is by definition) and also if any circuit $p \rightleftarrows q$ is impossible. Hence

$$\boxed{p\,\mathcal{J}(q) \le 0 \text{ and } q\,\mathcal{J}(p) \le 0 \Rightarrow \mathcal{J}(p) = \mathcal{J}(q)}$$ WARP

Lemma 2 : $\boxed{\mathcal{J} \text{ rational implies } \mathcal{J} \text{ satisfies WARP}}$

Proof: Let (C, e) rationalize $\mathcal{J}$, and let p, q satisfy $p\,\mathcal{J}(q) \le 0$, $q\,\mathcal{J}(p) \le 0$. Then $p(\mathcal{J}(q) + e) \le pe$, and $q(\mathcal{J}(p) + e) \le qe$. Hence

Consumer problem at p

$$\max C$$

$$py \le pe$$

$$y \geq 0.$$

Unique solution: $\mathcal{J}(p) + e$

Feasible point: $\mathcal{J}(q) + e$

Hence

$$(\mathcal{J}(p) + e)\, C\, (\mathcal{J}(q) + e)$$

Consumer problem at q

$$\max C$$

$$qy \le qe$$

$$y \geq 0$$

Unique solution $\mathcal{J}(q) + e$

Feasible point $\mathcal{J}(p) + e$

Hence

$$(\mathcal{J}(q) + e)\, C\, (\mathcal{J}(p) + e)$$

C is indifferent between $\mathcal{J}(p) + e$ and $\mathcal{J}(q) + e$

$\mathcal{J}(p)+e$ and $\mathcal{J}(q)+e$

$\downarrow$

$\mathcal{J}(p)+e = \mathcal{J}(q)+e$, by uniqueness of maximizers

$\downarrow$

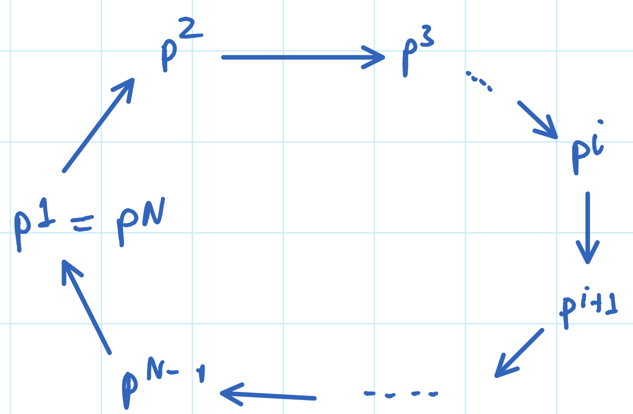
$\mathcal{J}(p) = \mathcal{J}(q)$.

STRONG AXIOM OF REVEALED PREFERENCE SARP

$\mathcal{J}$ satisfies SARP if the induced relation $\rightarrow$ on prices has no cycles of any finite length

Lemma 3: $\boxed{\mathcal{J} \text{ rational implies } \mathcal{J} \text{ satisfies SARP}}$

Proof: By lemma 2, the induced relation $\rightarrow$ on prices has no cycles of length 0, 1. Suppose, for contradiction, there is a cycle of length N ie



Let $x^i = \mathcal{J}(p^i)$. Then $x^i \neq x^{i+1}$ $p^i x^{i+1} \leq 0$ $\forall i = 1 .. N-1$

Let (C, e) rationalize $\mathcal{Y}$. Then for each i

$$(\mathcal{M}^i) \quad \begin{array}{l} \max C \\ p^i y \leq p^i e \\ y \geq 0 \end{array} \quad \begin{array}{|ll} \text{Unique solution} & y^i = x^i + e \\ \text{Feasible point} & y^{i+1} = x^{i+1} + e \\ \hline & y^i \, C \, y^{i+1} \end{array}$$

Since $y^1 = y^N$, we obtain that all y^i are equivalent for C.
Hence each y^i solves $(\mathcal{M}^i)$. By uniqueness, $y^i = y^1 \; \forall i$
ie $x^i + e = x^1 + e \;\; \forall i$, ie $x^i = x^1 \;\; \forall i$, contradiction

Is this all? Yes

Theorem: $\boxed{\mathcal{Y} : \mathbb{R}^L_{++} \to \mathbb{R}^L \text{ is rational iff it satisfies H- W- B- SARP}}$

Proof: We have already shown that if $\mathcal{Y}$ is rational then it satisfies H- W- B- SARP.

Suppose $\mathcal{Y}$ satisfies H- W- B- SARP. We construct a consumer (C, e) that rationalizes $\mathcal{Y}$.

$e = b$ (the lower bound of $\mathcal{Y}$)

For any two $x, y \in \mathbb{R}^L_+$ define a new relation $\Rightarrow$

$x \Rightarrow y$ iff

AND:

$x \neq y$

$\exists p,\ x = J(p) + e,\ py \leq pe$

ie for some p, x is the choice and y is feasible

$\Rightarrow$ has no cycles of length 0, since $x \Rightarrow x$ is never true

We show that $\Rightarrow$ has no cycles of any finite length

Suppose, for contradiction, that $\Rightarrow$ has a cycle of length N namely

$$x^1 = x^N \Rightarrow x^2 \Rightarrow x^3 \Rightarrow \cdots \Rightarrow x^i \Rightarrow x^{i+1} \Rightarrow \cdots \Rightarrow x^{N-1} \Rightarrow x^1 = x^N$$

Then there exist prices $p^1, p^2, \ldots, p^N$ such that

$x^i \neq x^{i+1}$

$x^i = J(p^i) + e$

$p^i x^{i+1} \leq p^i e$

$x^1 = x^N$

$J(p^i) \neq J(p^{i+1})$

$p^i J(p^{i+1}) \leq 0$

$J(p^1) = J(p^N)$

$p^1 \to p^2 \to \cdots \to p^{N-1} \to p^1$

contradiction to SARP ..

By Zorn's lemma, $\Rightarrow$ extends to a complete and transitive relation $\overset{*}{\Rightarrow}$ with no cycles, ie

$x \Rightarrow y$ implies $x \overset{*}{\Rightarrow} y$

$\overset{*}{\Rightarrow}$ is acyclic, complete, and transitive

We define C by

$x C y$ if $x = y$ or $x \overset{*}{\Rightarrow} y$

C is also complete and transitive. We now show that (C, e) rationalizes C, ie that for each $p >> 0$ the problem $\max C$, subject to $py \leq pe$, $y \geq 0$ is uniquely solved by $\hat{y} = J(p) + e$

By W $\quad p\hat{y} = pJ(p) + pe = pe \;\Big|\Rightarrow \hat{y}$ is feasible

By B $\quad \hat{y} >> 0$

Let $y \neq J(p) + e$ be any feasible point, ie $py \leq pe$, $y \geq 0$.

Then $\hat{y} \Rightarrow y$, hence $\hat{y} C y$, ie $\hat{y}$ is a maximizer.

We now show that $\hat{y}$ is the unique maximizer. Suppose $y \neq \hat{y}$ is another. Then $\hat{y} C y$ and $y C \hat{y}$. Since $y \neq \hat{y}$ we have

$y \overset{*}{\Rightarrow} \hat{y}$ and $\hat{y} \overset{*}{\Rightarrow} y$, contradicting acyclicity of $\overset{*}{\Rightarrow}$