The 2nd welfare theorem under asymmetric information

EXAMPLE 1

Consider an economy with two players, A and B, and two goods, consumption x and labor e.

$$\textbf{preferences } u_A = ax_A - \frac{1}{2}e_A^{\,2}, 0 < a \le 1, u_B = bx_B - \frac{1}{2}e_B^{\,2}, 0 < b \le 1$$

$$\textbf{production function} \quad x = e$$

endowments	x	e
A	0	1
B	0	1

Competitive equilibria

$$x_A = e_A = a \qquad u_A = \frac{a^2}{2}$$

$$x_B = e_B = b \qquad u_B = \frac{b^2}{2}$$

Pareto points

LEFT PART: $-0.5 \le \bar{u}_B \le -0.5b^2$

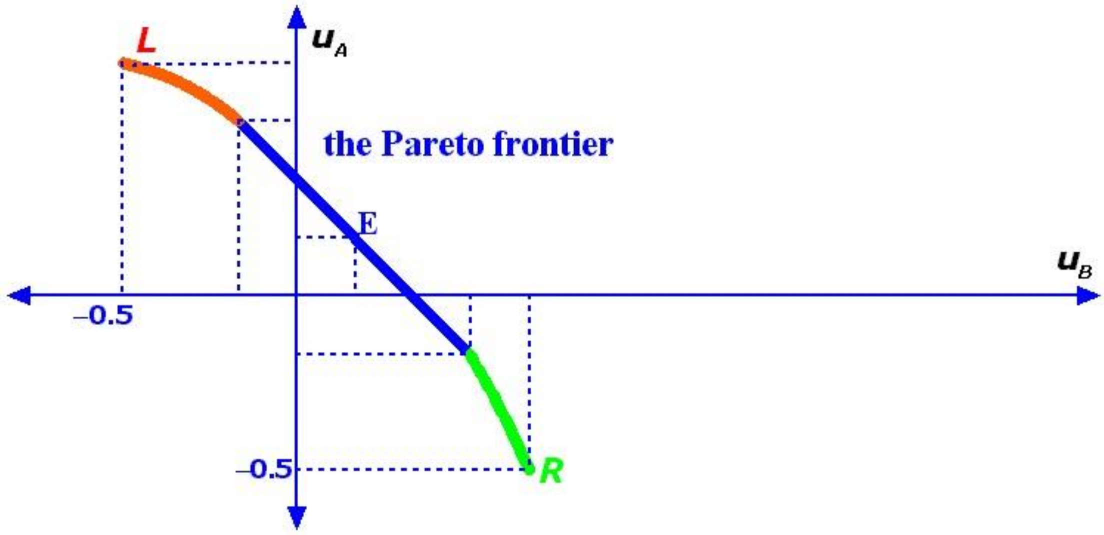
$$x_B = 0$$
$$e_B \ge b$$
$$x_A = e_B + a$$
$$e_A = a$$
$$u_A = 0.5a^2 + a\sqrt{-2u_B}$$
$$u_B = \bar{u}_B$$

MIDDLE PART: $-0.5b^2 \le \bar{u}_B \le ab + 0.5b^2$

$$e_A = a$$
$$e_B = b$$
$$x_A + x_B = a + b$$
$$u_A = 0.5a^2 + 0.5ab - \frac{a}{b}u_B$$
$$u_B = \bar{u}_B$$

RIGHT PART: $ab + 0.5b^2 \le \bar{u}_B \le b + 0.5b^2$

$$x_A = 0$$
$$e_A \ge a$$
$$x_B = e_A + b$$
$$e_B = b$$
$$u_A = -0.5\left(\frac{\bar{u}_B - 0.5b^2}{b}\right)^2$$
$$u_B = \bar{u}_B$$



Implementing Pareto points under asymmetric information

The implementing authorities know that there are two agents named A and B; the endowment of each agent; and that each agent can have any utility function of the form

$$hx - \frac{1}{2}e^2 \qquad 0 < h \leq 1$$

. In other words, the type of each agent is the value of the preference parameter h.

The direct revelation mechanism

- *Agent A declares a type* $\hat{a} \in (0,1]$
- *Agent B declares a type* $\hat{b} \in (0,1]$
- *The mechanism determines the consumptions* $x_i(\hat{a},\hat{b})$ *and the labor supplies* $e_i(\hat{a},\hat{b})$ *of each agent, and therefore defines*
- *a game with payoff functions*

$$u_A(a,\hat{a},b,\hat{b}) = ax_A(\hat{a},\hat{b}) - \frac{1}{2}e_A^2(\hat{a},\hat{b})$$

$$u_B(a,\hat{a},b,\hat{b}) = bx_B(\hat{a},\hat{b}) - \frac{1}{2}e_B^2(\hat{a},\hat{b})$$

- *We will use anonymous mechanisms, namely mechanisms that make decisions based solely on announced types and not on the players names*

$$x_A\left(\hat{a},\hat{b}\right)=x_B\left(\hat{b},\hat{a}\right)$$
$$e_A\left(\hat{a},\hat{b}\right)=e_B\left(\hat{b},\hat{a}\right)$$

Implementing the middle part of the Pareto frontier

We impose on the mechanism the property of Pareto points on the middle part of the frontier, namely

$$x_A\left(\hat{a},\hat{b}\right)+x_B\left(\hat{a},\hat{b}\right)=\hat{a}+\hat{b}$$

$$e_A\left(\hat{a},\hat{b}\right)=\hat{a}$$

$$e_B\left(\hat{a},\hat{b}\right)=\hat{b}$$

The game defined by such a mechanism then becomes

$$u_A\left(a,\hat{a},b,\hat{b}\right)=ax_A\left(\hat{a},\hat{b}\right)-\frac{1}{2}\hat{a}^2$$
$$u_B\left(a,\hat{a},b,\hat{b}\right)=b\left[\hat{a}+\hat{b}-x_A\left(\hat{a},\hat{b}\right)\right]-\frac{1}{2}\hat{b}^2$$

We now write the necessary conditions for truthful revelation of preferences to be a Nash equilibrium

$$\left.\frac{\partial u_A\left(a,\hat{a},b,b\right)}{\partial\hat{a}}\right|_{\hat{a}=a}=0,\ \forall a\in(0,1)$$

$$\left.\frac{\partial u_B\left(a,a,b,\hat{b}\right)}{\partial\hat{b}}\right|_{\hat{b}=b}=0,\ \forall b\in(0,1)$$

These conditions amount to

$$\frac{\partial x_A(a,b)}{\partial a} = 1 \qquad (1)$$

$$\frac{\partial x_A(a,b)}{\partial b} = 0 \qquad (2)$$

Solving (1), we obtain

$$x_A(a,b) = x_A(0,b) + a \qquad (1')$$

Solving (2), we obtain

$$x_A(a,b) = x_A(a,0) \qquad (2')$$

Equating the right-hand sides of the last two equations, we obtain

$$x_A(a,0) - a = x_A(0,b) \quad \forall a \in (0,1) \forall\ b \in (0,1) \qquad (3)$$

The right-hand side of (3) is independent of a, hence so is the left-hand side. Hence

$$\frac{\partial x_A(a,0)}{\partial a} - 1 = 0$$

Solving this we obtain

$$x_A(a,0) = x_A(0,0) + a \qquad (4)$$

By (2′) and (4)

$$x_A(a,b) = x_A(0,0) + a \qquad (5)$$

We know that

$$x_A(a,b) + x_B(a,b) = a + b \qquad (6)$$

and that ,therefore,

$$0 \le x_A(a,b) \le a + b \qquad (7)$$

By 5,7

$$0 \le x_A(0,0) \le b, \forall b \in (0,1)$$

which implies

$$x_A(0,0)=0 \quad (8)$$

By 5,8,6

$$\begin{aligned} x_A(a,b) &= a = e_A(a,b) \\ x_B(a,b) &= b = e_B(a,b) \end{aligned} \quad (9)$$

The anonymity condition is satisfied. Hence the only implementable point from the middle part of the Pareto frontier is the competitive equilibrium point E.

Implementing the left part of the Pareto frontier

We impose on the mechanism the property of Pareto points on the left part of the frontier, namely

$$\begin{aligned} e_A(\hat{a},\hat{b}) &= \hat{a} \\ x_B(\hat{a},\hat{b}) &= 0 \\ x_A(\hat{a},\hat{b}) &= \hat{a} + e_B(\hat{a},\hat{b}) \end{aligned} \quad (10)$$

$$e_B(\hat{a},\hat{b}) \geq \hat{b} \quad (11)$$

We use the equational part to write down the game defined by this mechanism

$$\begin{aligned} u_A(a,\hat{a},b,\hat{b}) &= a\left(\hat{a} + e_B(\hat{a},\hat{b})\right) - \frac{1}{2}\hat{a}^2 \\ u_B(a,\hat{a},b,\hat{b}) &= -\frac{1}{2}e_B^{\,2}(\hat{a},\hat{b}) \end{aligned}$$

We now write the necessary conditions for truthful revelation of preferences to be a Nash equilibrium

$$\left. \frac{\partial u_A\left(a,\hat{a},b,b\right)}{\partial \hat{a}} \right|_{\hat{a}=a} = 0, \forall a \in (0,1)$$

$$\left. \frac{\partial u_B\left(a,a,b,\hat{b}\right)}{\partial \hat{b}} \right|_{\hat{b}=b} = 0, \forall b \in (0,1)$$

These conditions and 11 amount to

$$\frac{\partial e_B(a,b)}{\partial a} = 0, \forall a \forall b$$

$$\frac{\partial e_B(a,b)}{\partial b} = 0, \forall a \forall b$$

By these equations and 11, $e_B(a,b) = e_B(0,0) = 1$

The mechanism we have derived is given by

$$\boxed{\begin{aligned} e_A(a,b) &= a \\ e_B(a,b) &= 1 \\ x_B(a,b) &= 0 \\ x_A(a,b) &= a+1 \end{aligned}}$$

The anonymity condition is not satisfied. Hence there is no implementable point on the left part of the Pareto frontier. If we don't impose anonymity, then the only implementable point on the left part of the frontier is the leftmost point L

Implementing the right part of the Pareto frontier

We impose on the mechanism the property of Pareto points on the left part of the frontier, namely

$$\boxed{\begin{aligned} x_A\left(\hat{a},\hat{b}\right) &= 0 \\ e_B\left(\hat{a},\hat{b}\right) &= \hat{b} \\ x_B\left(\hat{a},\hat{b}\right) &= e_A\left(\hat{a},\hat{b}\right) + \hat{b} \end{aligned}} \quad (12)$$

$$\boxed{e_A\left(\hat{a},\hat{b}\right)\geq \hat{a}} \quad (13)$$

We use the equational part to write down the game defined by this mechanism

$$\boxed{\begin{aligned} u_A\left(a,\hat{a},b,\hat{b}\right) &= -\frac{1}{2}e_A^{\,2}\left(\hat{a},\hat{b}\right) \\ u_B\left(a,\hat{a},b,\hat{b}\right) &= b\left(e_A\left(\hat{a},\hat{b}\right)+\hat{b}\right)-\frac{1}{2}\hat{b}^2 \end{aligned}}$$

We now write the necessary conditions for truthful revelation of preferences to be a Nash equilibrium

$$\left.\frac{\partial u_A\left(a,\hat{a},b,b\right)}{\partial \hat{a}}\right|_{\hat{a}=a} = 0, \forall a\in(0,1)$$

$$\left.\frac{\partial u_B\left(a,a,b,\hat{b}\right)}{\partial \hat{b}}\right|_{\hat{b}=b} = 0, \forall b\in(0,1)$$

These conditions and 13 imply

$$\frac{\partial e_A(a,b)}{\partial a} = 0, \forall a \forall b$$

$$\frac{\partial e_A(a,b)}{\partial b} = 0, \forall a \forall b$$

By these and 13,

$e_A(a,b) = e_A(0,0) = 1$. *The mechanism we have derived is*

$$\boxed{\begin{aligned} e_A(a,b) &= 1 \\ e_B(a,b) &= b \\ x_A(a,b) &= 0 \\ x_B(a,b) &= 1+b \end{aligned}}$$

The anonymity condition is not satisfied. Hence there is no implementable point on the right part of the Pareto frontier. If anonymity is not imposed, the rightmost point R is the only implementable point on the right part of the frontier

CONCLUSION: THE ONLY PARETO POINT IMPLEMENTABLE BY ANONYMOUS MECHANISMS IS COMPETITIVE EQUILIBRIUM

EXAMPLE 2

*We consider the same economy, with B **HAVING NO PRIVATE INFORMATION**, in the sense that everybody knows that **b=1**.Note that now the left part of the frontier is just the single point L, which is also on the middle part.*

The direct revelation mechanism

- *Agent A declares a type* $\hat{a} \in (0,1]$
- *The mechanism determines the consumptions* $x_i(\hat{a})$ *and the labor supplies* $e_i(\hat{a})$ *of each agent, and therefore defines*
- *a game with payoff functions*

$$u_A(a,\hat{a}) = ax_A(\hat{a}) - \frac{1}{2}e_A{}^2(\hat{a})$$

$$u_B(\hat{a}) = x_B(\hat{a}) - \frac{1}{2}e_B{}^2(\hat{a})$$

- *The anonymity condition is now*

$$e_A(1) = e_B(1)$$

$$x_A(1) = x_B(1)$$

Implementing the middle part of the Pareto frontier

We impose on the mechanism the property of Pareto points on the middle part of the

$$x_A(\hat{a}) + x_B(\hat{a}) = \hat{a} + 1$$

frontier, namely $e_A(\hat{a}) = \hat{a}$

$$e_B(\hat{a}) = 1$$

The game defined by such a mechanism then becomes

$$u_A(a,\hat{a}) = a x_A(\hat{a}) - \frac{1}{2}\hat{a}^2$$

$$u_B(\hat{a}) = \left[\hat{a} + 1 - x_A(\hat{a})\right] - \frac{1}{2}$$

We now write the necessary conditions for truthful revelation of preferences to be a Nash equilibrium

$$\left.\frac{\partial u_A(a,\hat{a})}{\partial \hat{a}}\right|_{\hat{a}=a} = 0, \forall a \in (0,1)$$

These conditions amount to

$$\frac{\partial x_A(a)}{\partial a} = 1 \qquad (1)$$

Solving (1), we obtain

$$x_A(a) = x_A(0) + a \qquad (2)$$

Now using the relation $x_A(a) + x_B(a) = a + 1$ *and 2 we obtain*

$$\begin{array}{l} x_A(a) = a + x_A(0),\ e_A(a) = a \\ x_B(a) = 1 - x_A(0), e_B(a) = 1 \\ 0 \le x_A(0) \le 1 \\ u_A = a x_A(0) + 0.5a^2 \\ u_B = 0.5 - x_A(0) \end{array} \qquad (3)$$

Imposing the anonymity condition we obtain

$$x_A(a) = a,\ e_A(a) = a$$
$$x_B(a) = 1, e_B(a) = 1$$
$$x_A(0) = 0$$
$$u_A = 0.5a^2$$
$$u_B = 0.5$$

The corresponding Pareto frontier, without the anonymity condition, is

$$u_A = 0.5a + 0.5a^2 - au_B \ ,\ -0.5 \le u_B \le 0.5$$

while in the full information frontier we have $-0.5 \le u_B \le 0.5 + a$

Hence the only implementable points from the middle part of the Pareto frontier are those that correspond to utilities that satisfy $-0.5 \le u_B \le 0.5$. *The missing points are those that satisfy* $1 \le x_B(a) \le 1 + a$.

On the other hand, with the anonymity condition, the only implementable point on the middle part of the frontier is competitive equilibrium.

Implementing the right part of the Pareto frontier

We impose on the mechanism the property of Pareto points on the right part of the frontier, namely

$$\boxed{\begin{aligned} x_A(\hat{a}) &= 0 \\ e_B(\hat{a}) &= 1 \\ x_B(\hat{a}) &= e_A(\hat{a}) + 1 \end{aligned}} \quad (4) \quad \boxed{e_A(\hat{a}) \ge \hat{a}} \quad (5)$$

We use the equational part to write down the game defined by this mechanism

$$u_A(a, \hat{a}) = -\frac{1}{2} e_A{}^2(\hat{a})$$
$$u_B(\hat{a}) = e_A(\hat{a}) + \frac{1}{2}$$

We now write the necessary conditions for truthful revelation of preferences to be a Nash equilibrium

$$\left.\frac{\partial u_A(a,\hat{a})}{\partial \hat{a}}\right|_{\hat{a}=a} = 0, \forall a \in (0,1)$$

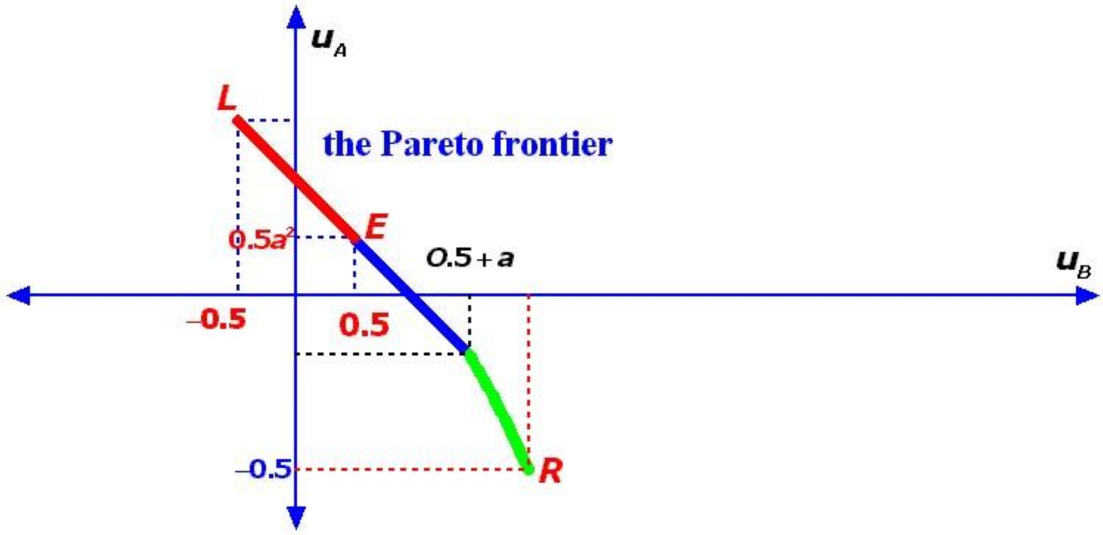
These conditions and 5 imply

$$\frac{\partial e_A(a)}{\partial a} = 0, \forall a$$

By these and 5, $e_A(a) = e_A(0) = 1$. *The mechanism we have derived is*

$$\boxed{\begin{array}{l} e_A(a) = 1 \\ e_B(a) = 1 \\ x_A(a) = 0 \\ x_B(a) = 2 \end{array}}$$

The utilities are given by $u_A = -0.5, u_B = 1.5$ *This mechanism does not satisfy anonymity.Hence, without anonymity, the only implementable point from the right part of the Pareto frontier is the rightmost point L.In the picture, the implementable part of the frontier without anonymity is in red*



On the other hand, the only point implementable by anonymous mechanisms is competitive equilibrium E